Inflation
A Review

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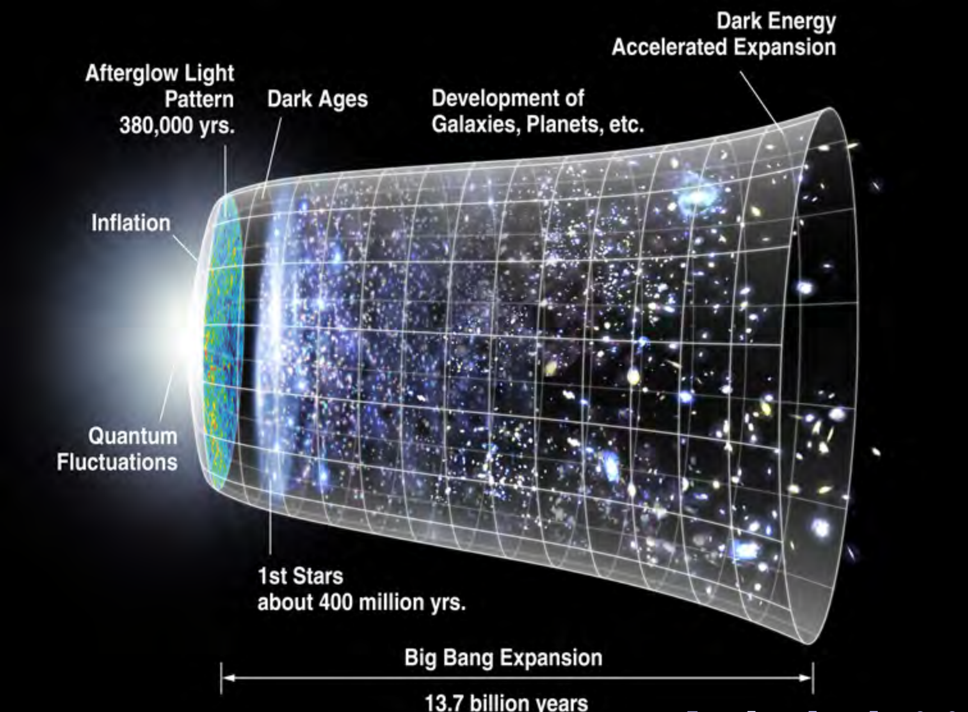
KBFI,

December 15, 2014

Planck Collaboration


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
380,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

3 Cosmology

A homogenous and isotropic universe is described by the Friedmann-Lemaître-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

Matter is a perfect fluid:

$$T^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)$$

4 Cosmology

Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_P^2}\rho - \frac{k}{a^2}$$
$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p) = -\frac{1}{6M_P^2}\rho(1 + 3w),$$

where the reduced Planck mass
$M_P^2 = \dfrac{1}{8\pi G} = 2.4 \times 10^{18}$ GeV

Continuity equation

$$\dot{\rho} = -3H(\rho + p) = -3H\rho(1 + w)$$

5 Critical Density

From the Friedmann equation,

$$k = a^2 \left(\frac{1}{3M_P^2} - H^2 \right),$$

thus $k = 0$ if

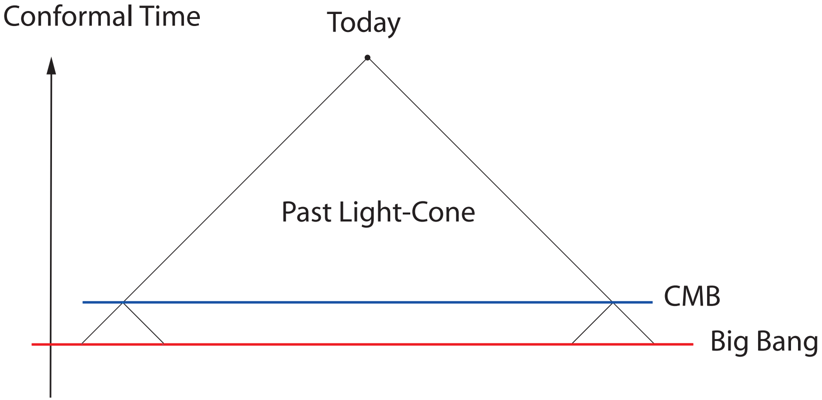
$$\rho = \rho_c = 3M_P^2 H^2$$

or $\Omega = 1$, where the density parameter

$$\Omega = \frac{\rho}{\rho_c} = \frac{1}{3M_P^2} \frac{\rho}{H^2} \tag{1}$$

6 Problems of Big Bang Cosmology

- Why is the Universe homogenous?



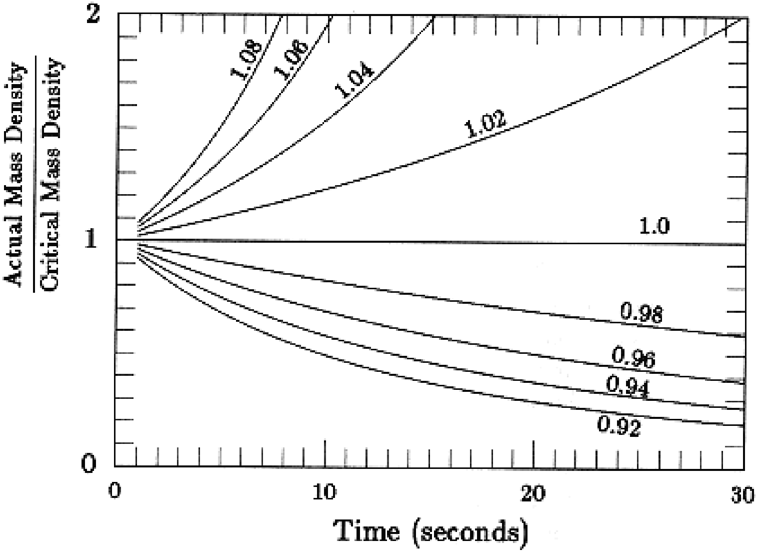
Problems of Big Bang Cosmology

- Why is the Universe homogenous?
- Why is the Universe flat?

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\ln a} = (1+3w)\Omega(\Omega-1)$$

- $\Omega = 1$ is an unstable fixed point for $1+3w > 0$
- $|\Omega - 1| \leqslant \mathcal{O}(10^{-61})$ at Planck time

Problems of Big Bang Cosmology



9 Problems of Big Bang Cosmology

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- Unlikely *initial conditions*

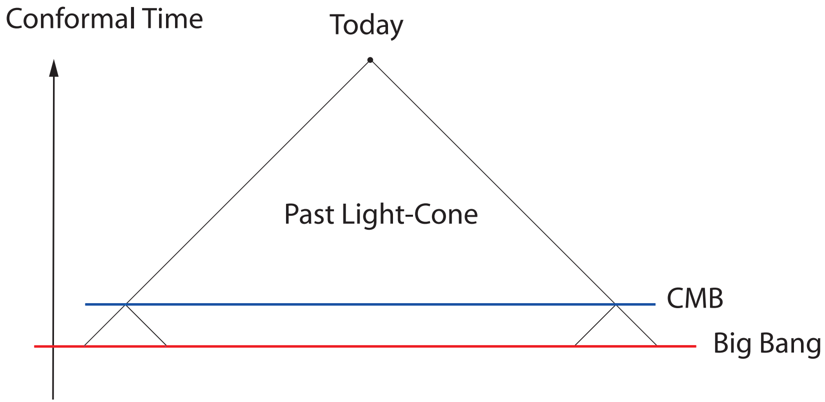
10 Vacuum Energy & Inflation

- For $p_\Lambda = -\rho_\Lambda$ or $w = -1$, one has $\dot{\rho} = 0$
- The scale factor grows as

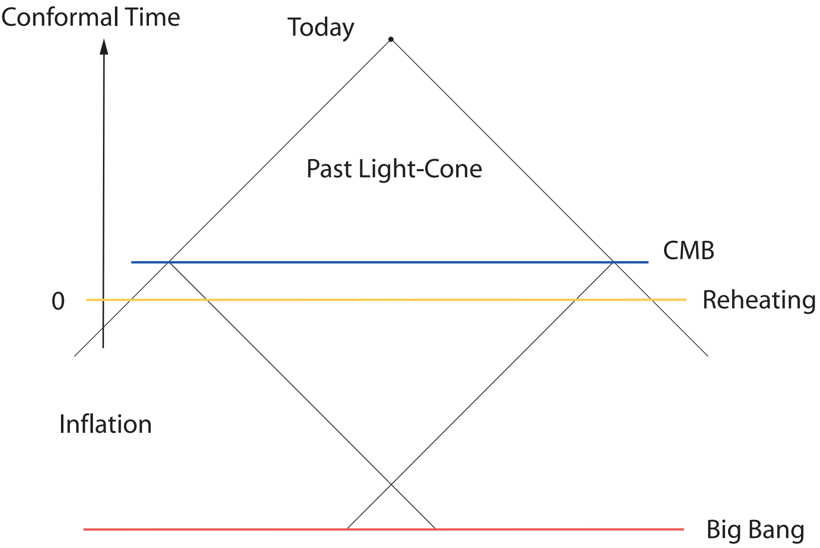
$$a(t) \propto e^{Ht},$$

where $H = \sqrt{\frac{\rho_\Lambda}{3M_P^2}}$

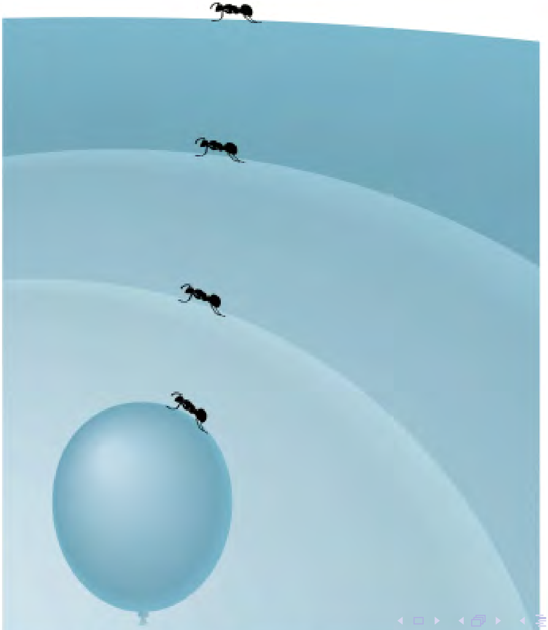
11 Problems of Big Bang Cosmology



12 Problems of Big Bang Cosmology



13 Problems of Big Bang Cosmology



Inflaton Density as Vacuum Energy

Action for gravity and scalar field ϕ

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$$

Energy density & pressure

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$
$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

- $w = \frac{p}{\rho} \approx -1$ if $\dot{\phi} \ll V(\phi)$

15 Inflaton Density as Vacuum Energy

Equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + V(\phi)' = 0$$

Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

16 Slow-Roll Conditions

Equation of motion:

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Friedmann equation

$$H^2 = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

- $\dot{\phi}^2 \ll V(\phi)$
- $|\ddot{\phi}| \ll |3H\dot{\phi}|, |V(\phi)'|$

17 Slow-Roll Conditions

The slow-roll parameters are

$$\epsilon(\phi) = \frac{1}{2} M_P^2 \frac{V'(\phi)}{V(\phi)}$$
$$\eta(\phi) = M_P^2 \frac{V''(\phi)}{V(\phi)}$$

For slow-roll it is necessary to have $\epsilon \ll 1$ and $\eta \ll 1$

18 Slow-Roll Inflation

Friedmann equations give

$$3H\dot{\phi} \approx -V(\phi)'$$
$$H^2 \approx \frac{V(\phi)}{3M_P^2}$$

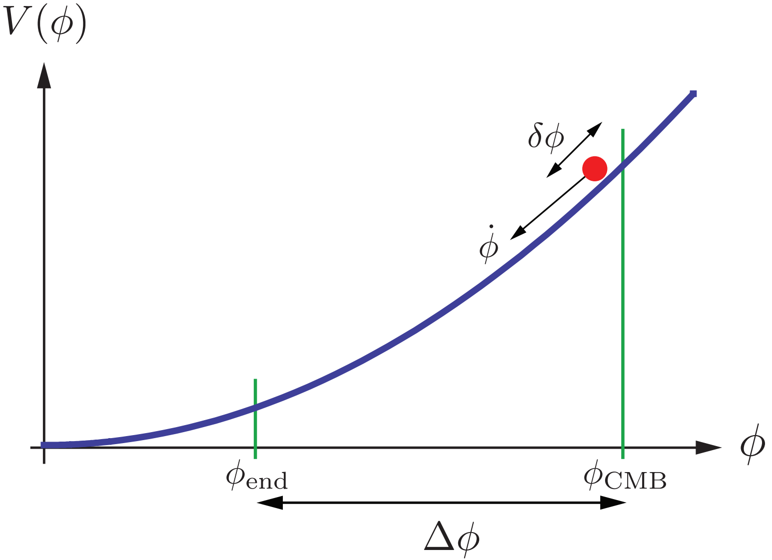
and

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3p) = H^2(1-\epsilon),$$

where

$$\epsilon \equiv \frac{3}{2}(w+1) = \frac{1}{2M_P^2}\frac{\dot{\phi}^2}{H^2} \approx \frac{1}{2}M_P^2\frac{V'(\phi)}{V(\phi)}$$

Slow-Roll Inflation



Number of e-Folds

The number of e-folds is given by

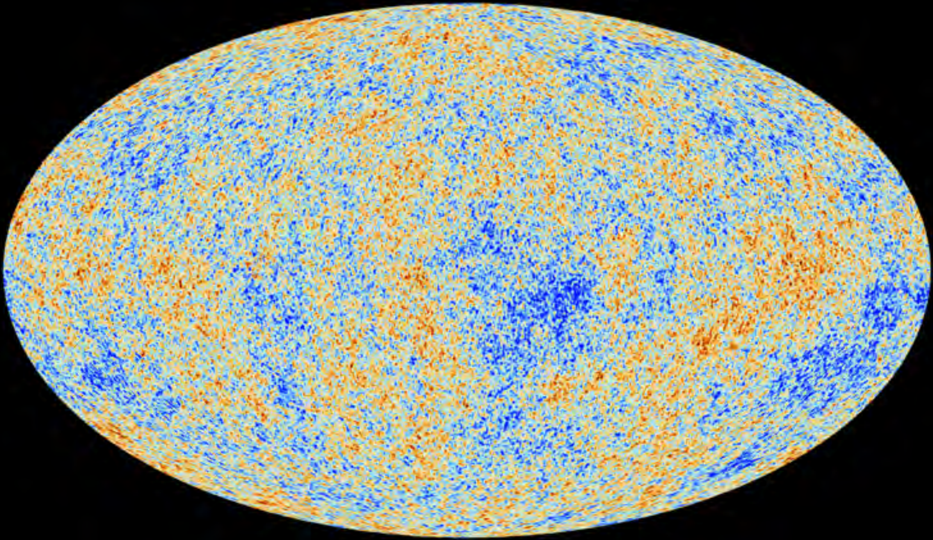
$$N_* = \int_{t_e}^{t_*} dtH = \int_{t_e}^{t_*} d\phi \frac{H}{\dot{\phi}} \approx \frac{1}{M_P^2} \int_{\phi_e}^{\phi_*} d\phi \frac{V(\phi)'}{V(\phi)}$$

- Minimal required number of e-folds is $N = 50 \ldots 60$

21 Minimal Required Number of *e*-Folds

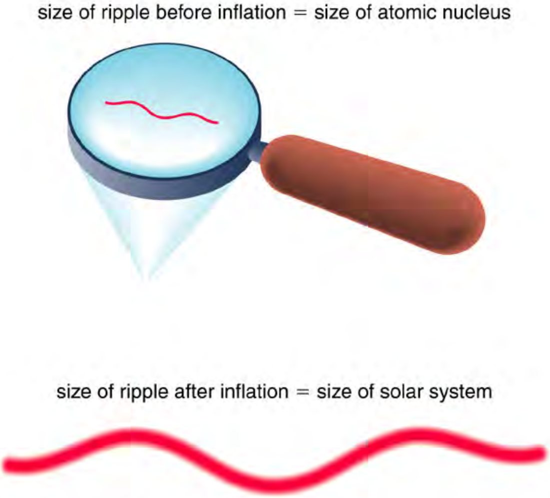
"I'll tell you what's beyond the observable universe -- lots and lots of unobservable universe."

22 Density Variations



Planck Collaboration

23 Quantum Fluctuations



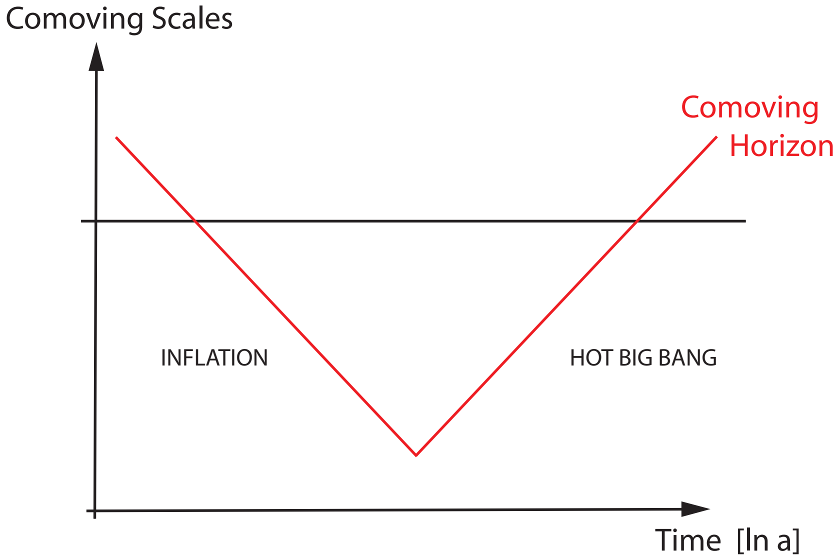
24 Quantum Fluctuations

- Classical field and fluctuation: $\phi \to \phi + \varphi$
- φ is approximately free quantum field
- For Fourier components of φ,

$$\ddot{\varphi}_k + 3H\dot{\varphi}_k + \frac{k^2}{a^2}\varphi_k = 0,$$

  where $\frac{k}{a} \sim \lambda^{-1}$
- For $\frac{a}{k} \ll H^{-1}$, the field behaves as Klein-Gordon
- For $\frac{a}{k} \gg H^{-1}$, there is overdamping and the fluctuation 'freezes'

25 Quantum Fluctuations



Power Spectra

The power spectra of scalar and tensor perturbations are parametrised by

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k^*}\right)^{n_s - 1 + \frac{1}{2}\frac{dn_s}{d\ln k}\ln\frac{k}{k^*} + \ldots}$$

$$\mathcal{P}_t(k) = A_t \left(\frac{k}{k^*}\right)^{n_t - 1 + \frac{1}{2}\frac{dn_t}{d\ln k}\ln\frac{k}{k^*} + \ldots}$$

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In the slow-roll approximation

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{2\pi}\right)_{k=aH}$$

$$\mathcal{P}_t(k) = \left(\frac{16H^2}{\pi M_P^2}\right)_{k=aH}$$

27 Power Spectra

The power spectra of scalar and tensor (gravity wave) perturbations are parametrised by

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In the slow-roll approximation

$$A_s \approx \frac{V}{24\pi^2 M_P^4 \epsilon}$$

$$n_s \approx 1 - 6\epsilon + 2\eta$$

$$r = \frac{\mathcal{P}_{\mathcal{R}}(k)}{\mathcal{P}_t(k)} \approx 16\epsilon = -8n_t$$

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Recipe for Slow-Roll Inflation

- Calculate the field value at the end of inflation:

$$\epsilon(\phi_e) = \frac{M_P^2}{2}\left(\frac{V'(\phi_e)}{V(\phi_e)}\right)^2 = 1$$

- Calculate ϕ_* from the required number of e-folds N:

$$N = \frac{1}{M_P}\int_{\phi_e}^{\phi_*} \frac{d\varphi}{\sqrt{2\epsilon(\varphi)}}$$

Recipe for Slow-Roll Inflation

- Calculate r and n_s for e-folds $N = [50, 60]$:

$$r = 16\epsilon(\phi_*)$$
$$n_s = 1 - 6\epsilon(\phi_*) + 2\eta(\phi_*),$$

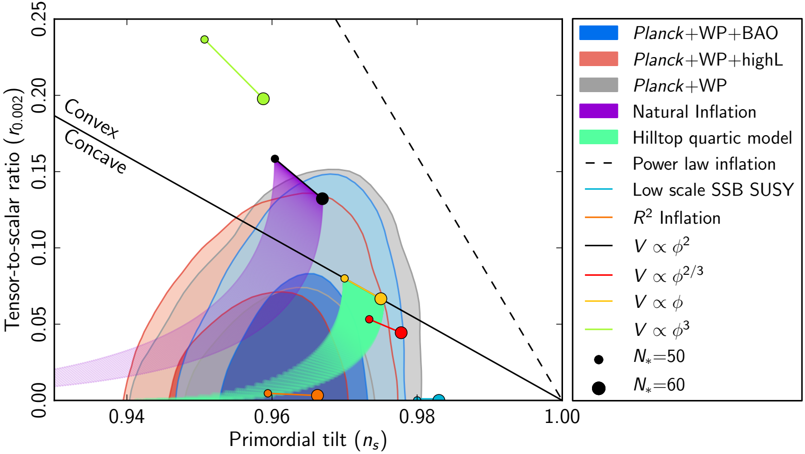
- From the amplitude of scalar perturbations

$$A_s^2 = \frac{1}{M_P^4}\frac{V(\phi_*)}{24\pi^2\epsilon(\phi_*)} \approx 2.45 \times 10^{-9}$$

fix the normalisation of V:

$$V(\phi_*) \approx (1.94 \times 10^{16}\ \mathrm{GeV})^4 \frac{r}{0.12}$$

Recipe for Slow-Roll Inflation



Planck: $n_s = 0.9603 \pm 0.0073$, $r < 0.12$

31 Monomial Chaotic Inflation

- Monomial potential

$$V = \alpha\phi^n$$

- The slow-roll parameters are

$$\epsilon(\phi) = \frac{1}{2}M_P^2\frac{V'(\phi)}{V(\phi)} = \frac{M_P^2}{2}\frac{n^2}{\phi^2}$$

$$\eta(\phi) = M_P^2\frac{V''(\phi)}{V(\phi)} = M_P^2\frac{(n-1)n}{\phi^2}$$

- $\epsilon(\phi_e) = 1$ gives

$$\phi_e = \frac{M_P}{\sqrt{2}}n$$

Monomial Chaotic Inflation

- The number of e-folds is

$$N = \frac{1}{M_P^2} \int_{\phi_*}^{\phi_e} d\phi \frac{V(\phi)'}{V(\phi)} = \frac{1}{2M_P^2} \frac{\phi_*^2 - \phi_e^2}{n},$$

which yields

$$\phi_* = \frac{M_P}{\sqrt{2}} \sqrt{n(4N+n)}$$

- The slow-roll parameters at the 'beginning' of inflation are

$$\epsilon(\phi) = \frac{n}{4N+n}$$

$$\eta(\phi) = \frac{2(n-1)}{4N+n}$$

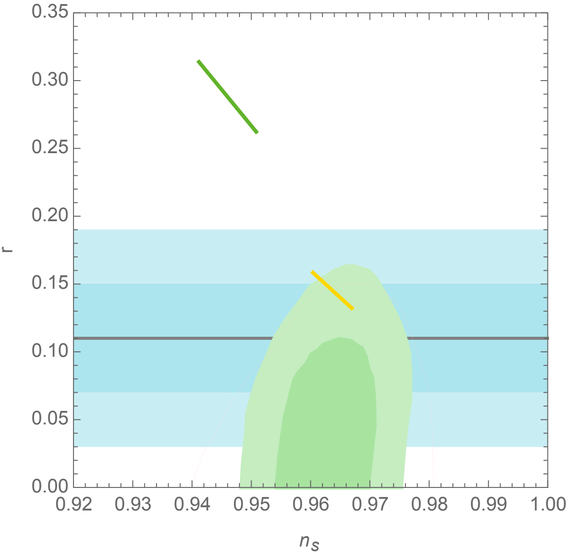
33 Monomial Chaotic Inflation

The tensor-to-scalar ratio and spectral tilt are

$$r = 16\epsilon(\phi_*) = \frac{16n}{4N+n}$$
$$n_s = 1 - 6\epsilon(\phi_*) + 2\eta(\phi_*) = 1 - \frac{2(2+n)}{4N+n)}$$

- For $n = 2$, $N = 50$, we have $r = 0.158$, $n_s = 0.9604$
- For $n = 4$, $N = 50$, we have $r = 0.314$, $n_s = 0.9412$

Monomial Chaotic Inflation



Colley, Gott 1409.4491; Cheng, Huang & Wang, 1409.7025

35 Monomial Chaotic Inflation

For

$$V = \alpha\phi^n,$$

normalisation of the spectrum

$$A_s \approx \frac{V(\phi_*)}{24\pi^2 M_P^4 \epsilon(\phi_*)} \approx \frac{2^{\frac{n}{2}} N (M_P \sqrt{Nn})^n}{6\pi^2 M_P^4 n} \alpha = 2.45 \times 10^{-9}$$

gives

- for $n = 2$, the inflaton mass is $m = 1.81 \times 10^{13}$ GeV
- for $n = 4$, the inflaton self-coupling is $\lambda = 6.83 \times 10^{-14}$

Other Possibilities

- Natural inflation, ...
- Non-minimal coupling to gravity
- Modified gravity ($f(R)$ theories)
- Non-canonical kinetic term
- Multi-field inflation

Lyth Bound

Lyth bound

$$\frac{\Delta\phi}{M_P} \approx \sqrt{\frac{r}{0.01}}$$

- $\Delta\phi > M_P$ for $r > 0.01$
- For $n = 2$, $\phi_* \approx 14M_P$
- For $n = 4$, $\phi_* \approx 20M_P$
- Problems with quantum gravity?

Conclusions

- Inflation solves the problems of Big Bang Cosmology
- Many possible models
- Controversy between Planck and BICEP2