

# EINSTEIN VERSUS JORDAN FRAME: $f(R)$ -THEORY

I. Jordan frame:

$$\mathcal{L} = f(R) + L_m. \quad (1)$$

Einstein frame:

$$\mathcal{L} = R/(2\kappa^2) + \text{scalar sector} + u(\phi)L_m. \quad (2)$$

Here, the matter sector is different from  $L_m$ , certain physical in-equivalence.

Example:

$$f(R) = \frac{R}{2\kappa^2} - \frac{c_1}{R^n}. \quad (3)$$

Its Einstein frame analog (at small curvature):

$$\mathcal{L} = \frac{R}{2\kappa^2} - \frac{g^{\mu\nu}\partial_\mu\phi\partial\phi_\nu}{2} - V(\phi), \quad V(\phi) \simeq e^{\text{const } \phi}. \quad (4)$$

Solutions:

$$H \sim \frac{1}{t}, \quad a_E \sim t^{3\left(\frac{n+1}{n+2}\right)^2}, \quad (5)$$

$$a_J \simeq t^{\pm\left(\frac{n+1}{n+2}\right)+3\left(\frac{n+1}{n+2}\right)^2}. \quad (6)$$

II. Initial conditions (for numerics) like stars.

III. Example (Einstein frame):

$$\mathcal{L} = \frac{R}{2\kappa^2} \mp \frac{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{2} - V(\phi). \quad (7)$$

After conformal transformation + equations of motion:

$$\mathcal{L} = e^{\pm i\kappa\sqrt{2/3}\phi} \frac{R}{2\kappa^2} - e^{\pm i\kappa\sqrt{2/3}\phi} V(\phi). \quad (8)$$

If  $V = V_0 e^{a\kappa\phi}$ ,

$$R = 2\kappa^2 V_0 \left(2 \mp i\sqrt{3/2}a\right) e^{\kappa\phi(R)\left(a \pm i\sqrt{2/3}\right)}. \quad (9)$$

Sometimes  $R$  is complex. Phantom scalar maybe mapped in complex  $f(R)$ .

IV. Example:

$$\mathcal{L} = \frac{R}{2\kappa^2} - \frac{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{2} - V_0 \cosh\left(\frac{2\phi}{\phi_0}\right). \quad (10)$$

Solution:

$$H \sim \frac{1}{t_0 - t} \rightarrow \text{Big Rip at } t = t_0. \quad (11)$$

At Big Rip,

$$F(R) \sim R^{2 - \frac{2/\phi_0}{\pm i\kappa\sqrt{2/3+2/\phi_0}}} , \quad (12)$$

becomes complex. Structure of singularity maybe changed too in explicit examples.

V. Perurbations: Gauge fixing problem.

VI. Quantum gravity: different parametrizations/variables.

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + cR\phi + V(\phi) . \quad (13)$$

One-loop EA (2d):

$$\mathcal{L} = -\frac{1}{2\epsilon} \left( 4R + \frac{2V}{c\phi} + \frac{2V'}{c} + \left( \frac{1}{\phi} - \frac{1}{c} \right) \Delta\phi - \frac{3}{\phi^2} \nabla\phi\nabla\phi \right) . \quad (14)$$

Field transformation:

$$\psi^2 = \frac{c}{\gamma} \phi, \quad g_{\mu\nu} \rightarrow e^{-2\rho} \tilde{g}_{\mu\nu},$$
$$\gamma > 0, \quad \rho = \frac{\gamma \psi^2}{4c^2} - \frac{1}{8\gamma} \log \psi. \quad (15)$$

Transformed action:

$$\mathcal{L} = \frac{1}{2} \partial\psi \partial\psi + \gamma \tilde{R}\psi^2 + U(\psi). \quad (16)$$

One-loop divergent EA is totally different.

On shell coincidence: using equations of motion.