Early-time and late-time accelerating universe from modified gravity:consistent unification.

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$F(R)$ gravity:General properties

The action of ghost-free $F(R)$ gravity

$$S_{F(R)} = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter}\right) . \tag{1}$$

The FRW equations in the Einstein gravity coupled with perfect fluid are given by

$$\rho_{\rm matter} = \frac{3}{\kappa^2}H^2 , \quad p_{\rm matter} = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot H\right) , \tag{2}$$

which allow us to define an effective equation of state (EoS) parameter as follows:

$$w_{\rm eff} = -1 - \frac{2\dot H}{3H^2} . \tag{3}$$

The field equation in the $F(R)$ gravity with matter is given by

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu\nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{{\rm matter}\,\mu\nu} . \tag{4}$$

By assuming a spatially flat FRW universe,

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1,2,3}\left(dx^i\right)^2 , \tag{5}$$

the equations corresponding to the FRW equations are given as follows:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot H\right)F'(R) - 18\left(4H^2\dot H + H\ddot H\right)F''(R) + \kappa^2\rho_{\rm matter} , \tag{6}$$

$$0 = \frac{F(R)}{2} - \left(\dot H + 3H^2\right)F'(R) + 6\left(8H^2\dot H + 4\dot H^2 + 6H\ddot H + \dddot H\right)F''(R)$$
$$+ 36\left(4H\dot H + \ddot H\right)^2 F'''(R) + \kappa^2 p_{\rm matter} . \tag{7}$$

$F(R)$ gravity: General properties

One can find several (often exact) solutions of (6). When we neglect the contribution from matter, by assuming that the Ricci tensor is covariantly constant, that is, $R_{\mu\nu} \propto g_{\mu\nu}$, Eq. (4) reduces to an algebraic equation:

$$0 = 2F(R) - RF'(R)\,. \tag{8}$$

If Eq. (8) has a solution, the (anti-)de Sitter, the Schwarzschild-(anti-)de Sitter space, and/or the Kerr-(anti-)de Sitter space is an exact vacuum solution.

Now we assume that $F(R)$ behaves as $F(R) \propto f_0 R^m$. Then Eq. (6) gives

$$\begin{aligned} 0 =& f_0 \left\{ -\frac{1}{2}\left(6\dot{H} + 12H^2\right)^m + 3m\left(\dot{H} + H^2\right)\left(6\dot{H} + 12H^2\right)^{m-1} \right. \\ & \left. -3mH\frac{d}{dt}\left\{\left(6\dot{H} + 12H^2\right)^{m-1}\right\}\right\} + \kappa^2 \rho_0 a^{-3(1+w)}\,. \end{aligned} \tag{9}$$

Eq. (7) is irrelevant because it can be derived from (9). When the contribution from the matter can be neglected ($\rho_0 = 0$), the following solution exists:

$$H \sim \frac{-\frac{(m-1)(2m-1)}{m-2}}{t}\,, \tag{10}$$

which corresponds to the following EoS parameter (3):

$$w_{\mathrm{eff}} = -\frac{6m^2 - 7m - 1}{3(m-1)(2m-1)}\,. \tag{11}$$

$F(R)$ gravity: General properties

On the other hand, when the matter with a constant EoS parameter w is included, an exact solution of (9) is given by

$$a = a_0 t^{h_0}\,, \quad h_0 \equiv \frac{2m}{3(1+w)}\,,$$

$$a_0 \equiv \left[-\frac{3f_0 h_0}{\kappa^2 \rho_0} \left(-6h_0 + 12h_0^2 \right)^{m-1} \left\{ (1-2m)(1-m) - (2-m)h_0 \right\} \right]^{-\frac{1}{3(1+w)}}\,, \tag{12}$$

and we find the effective EoS parameter (3) as

$$w_{\rm eff} = -1 + \frac{w+1}{m}\,. \tag{13}$$

These solutions (10) and (12) show that modified gravity may describe early/late-time universe acceleration.

$F(R)$ gravity: Scalar-tensor description

One can rewrite $F(R)$ gravity as the scalar-tensor theory. By introducing the auxiliary field A, the action (1) of the $F(R)$ gravity is rewritten in the following form:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right)+F(A)\right\} . \tag{14}$$

By the variation of A, one obtains $A=R$. Substituting $A=R$ into the action (14), one can reproduce the action in (1). Furthermore, by rescaling the metric as $g_{\mu\nu}\to \mathrm{e}^{\sigma}g_{\mu\nu}$ $\left(\sigma=-\ln F'(A)\right)$, we obtain the Einstein frame action:

$$\begin{aligned} S_E =& \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R-\frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) , \\ V(\sigma) =& \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} . \end{aligned} \tag{15}$$

Here $g\left(\mathrm{e}^{-\sigma}\right)$ is given by solving the equation $\sigma=-\ln\left(1+f'(A)\right)=-\ln F'(A)$ as $A=g\left(\mathrm{e}^{-\sigma}\right)$. Due to the conformal transformation, a coupling of the scalar field σ with usual matter arises. Since the mass of σ is given by

$$m_\sigma^2 \equiv \frac{3}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{\left(F'(A)\right)^2} + \frac{1}{F''(A)}\right\} , \tag{16}$$

unless m_σ is very large, the large correction to the Newton law appears.

$F(R)$ gravity: Viable modified gravities

As an example, we may consider the following exponential model

$$F(R) = R + \alpha \left(\mathrm{e}^{-bR} - 1 \right) . \tag{17}$$

Here α and b are constants. One can regard α as an effective cosmological constant and we choose the parameter b so that $1/b$ is much smaller than the curvature R_0 of the present universe. Then in the region $R \gg R_0$, we find

$$m_\sigma^2 \sim \frac{\mathrm{e}^{bR}}{2\alpha b^2} , \tag{18}$$

which is positive and m_σ^2 could be very large and the correction to the Newton law is very small. In paper by Hu-Sawicky 2007, the one of the first examples of "realistic" $F(R)$ model was proposed. Currently, several viable models are proposed.
In order to obtain a realistic and viable model, $F(R)$ gravity should satisfy the following conditions:

1. When $R \to 0$, the Einstein gravity is recovered, that is,
$$F(R) \to R \quad \text{that is,} \quad \frac{F(R)}{R^2} \to \frac{1}{R} . \tag{19}$$
This also means that there is a flat space solution.
2. There appears a stable de Sitter solution, which corresponds to the late-time acceleration and, therefore, the curvature is small $R \sim R_L \sim \left(10^{-33}\,\mathrm{eV}\right)^2$. This requires, when $R \sim R_L$,
$$\frac{F(R)}{R^2} = f_{0L} - f_{1L} \left(R - R_L\right)^{2n+2} + o\left(\left(R - R_L\right)^{2n+2}\right) . \tag{20}$$
Here, f_{0L} and f_{1L} are positive constants and n is a positive integer. Of course, in some cases this condition may not be strictly necessary.

$F(R)$ gravity: Viable modified gravities

3. There appears a quasi-stable de Sitter solution that corresponds to the inflation of the early universe and, therefore, the curvature is large $R \sim R_I \sim \left(10^{16\sim 19}\,\mathrm{GeV}\right)^2$. The de Sitter space should not be exactly stable so that the curvature decreases very slowly. It requires

$$\frac{F(R)}{R^2} = f_{0I} - f_{1I}\left(R - R_I\right)^{2m+1} + o\left(\left(R - R_I\right)^{2m+1}\right) . \qquad (21)$$

Here, f_{0I} and f_{1I} are positive constants and m is a positive integer.

4. In order to avoid the curvature singularity when $R \to \infty$, $F(R)$ should behaves as

$$F(R) \to f_\infty R^2 \quad \text{that is} \quad \frac{F(R)}{R^2} \to f_\infty . \qquad (22)$$

Here, f_∞ is a positive and sufficiently small constant. Instead of (22), we may take

$$F(R) \to f_{\tilde{\infty}} R^{2-\epsilon} \quad \text{that is} \quad \frac{F(R)}{R^2} \to \frac{f_{\tilde{\infty}}}{R^\epsilon} . \qquad (23)$$

Here, $f_{\tilde{\infty}}$ is a positive constant and $0 < \epsilon < 1$. The above condition (22) or (23) prevents both the future singularity and the singularity due to large density of matter.

5. To avoid the anti-gravity, we require

$$F'(R) > 0 , \qquad (24)$$

which is rewritten as

$$\frac{d}{dR}\left(\ln\left(\frac{F(R)}{R^2}\right)\right) - \frac{2}{R} . \qquad (25)$$

$F(R)$ gravity: Viable modified gravities

6. Combining conditions (19) and (24), one finds
$$F(R) > 0\,. \tag{26}$$

7. To avoid the matter instability (Dolgov-Kawasaki 2003), we require
$$\begin{aligned} U(R_b) \equiv & \frac{R_b}{3} - \frac{F^{(1)}(R_b)F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} - \frac{F^{(1)}(R_b)}{3F^{(2)}(R_b)} \\ & + \frac{2F(R_b)F^{(3)}(R_b)}{3F^{(2)}(R_b)^2} - \frac{F^{(3)}(R_b)R_b}{3F^{(2)}(R_b)^2} < 0\,. \end{aligned} \tag{27}$$

The conditions 1 and 2 tell that an extra, unstable de Sitter solution must appear at $R = R_e$ $(0 < R_e < R_L)$. Since the universe evolution will stop at $R = R_L$ because the de Sitter solution $R = R_L$ is stable; the curvature never becomes smaller than R_L and, therefore, the extra de Sitter solution is not realized.
An example of viable $F(R)$ gravity is given below
$$\begin{aligned} \frac{F(R)}{R^2} = & \left\{ \left(X_m(R_I;R) - X_m(R_I;R_1) \right) \left(X_m(R_I;R) - X_m(R_I;R_L) \right)^{2n+2} \right. \\ & \left. + X_m(R_I;R_1)\, X_m(R_I;R_L)^{2n+2} + f_\infty^{2n+3} \right\}^{\frac{1}{2n+3}}\,, \\ X_m(R_I;R) \equiv & \frac{(2m+1)\,R_I^{2m}}{(R-R_I)^{2m+1} + R_I^{2m+1}}\,. \end{aligned} \tag{28}$$

Here, n and m are integers greater or equal to unity, and $n, m \geq 1$ and R_1 is a parameter related with R_e by
$$X(R_I;R_e) = \frac{(2n+2)\,X(R_I;R_1)\,X(R_I;R_1) + X(R_I;R_L)}{2n+3}\,. \tag{29}$$

We also assume $0 < R_1 < R_L \ll R_I$.

$F(R)$ gravity: Viable modified gravities

Another realistic theory unifying inflation with dark energy is given in

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \qquad (30)$$

Here Λ is the effective cosmological constant in the present universe and we also assume the parameter R_0 is almost equal to Λ. R_i and Λ_i are typical values of the curvature and the effective cosmological constant. α is a constant: $1 < \alpha \leq 2$. Generalizations: coupling of curvature with trace of EMT (Harko-Lobo- -Nojiri-Odintsov) or with EMT (Saez-Gomez).

Mimetic F(R) gravity

This theory makes natural unification of inflation, late-time acceleration and dark matter via unique gravitational theory. Proposal of mimetic theory:Mukhanov-Chamseddine. In the mimetic model, we parametrize the metric in the following form.

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu} \,. \tag{31}$$

Instead of considering the variation of the action with respect to $g_{\mu\nu}$, we consider the variation with respect to $\hat{g}_{\mu\nu}$ and ϕ. Because the parametrization is invariant under the Weyl transformation $\hat{g}_{\mu\nu} \to \mathrm{e}^{\sigma(x)}\hat{g}_{\mu\nu}$, the variation over $\hat{g}_{\mu\nu}$ gives the traceless part of the equation. Proposal of mimetic F(R) gravity: Nojiri-Odintsov,arXiv:1408.3561. In case of $F(R)$ gravity, by using the parametrization of the metric as above,

$$S = \int d^4x \sqrt{-g\left(\hat{g}_{\mu\nu},\phi\right)}\left(F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) + \mathcal{L}_{\mathrm{matter}}\right) \,. \tag{32}$$

Mimetic F(R) gravity

Field equations have the following form:

$$\begin{aligned}
0 =&\frac{1}{2}g_{\mu\nu}F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) - R\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu} F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) \\
&+ \nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)_{\mu} \nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)_{\nu} F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) \\
&- g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu} \Box\left(\hat{g}_{\mu\nu},\phi\right) F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) + \frac{1}{2}T_{\mu\nu} \\
&+ \partial_\mu\phi\partial_\nu\phi\left(2F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) - R\left(\hat{g}_{\mu\nu},\phi\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\right. \\
&\left.-3\Box\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) + \frac{1}{2}T\right) ,
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
0 =&\nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)^{\mu}\left(\partial_\mu\phi\left(2F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) - R\left(\hat{g}_{\mu\nu},\phi\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\right.\right. \\
&\left.\left.-3\Box\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) + \frac{1}{2}T\right)\right) .
\end{aligned} \tag{34}$$

We should note that any solution of the standard $F(R)$ gravity is also a solution of the mimetic $F(R)$ gravity. This is because in the standard $F(R)$ gravity, Eqs. (33)–(34) are always satisfied since we find $2F(R) - RF'(R) - 3\Box F'(R) + \frac{1}{2}T = 0$. The mimetic $F(R)$ gravity is ghost-free and conformally invariant theory.

Mimetic F(R) gravity

FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} dx^{i^2}\,, \tag{35}$$

with $R = 6\dot{H} + 12H^2$ and ϕ is equal to t (due to mimetic form of metric).
Field equations: Eq. (34) gives

$$\begin{aligned}\frac{C_\phi}{a^3} =& 2F(R) - RF'(R) - 3\Box F'(R) + \frac{1}{2}T \\ =& 2F(R) - 6\left(\dot{H} + 2H^2\right)F'(R) + 3\frac{d^2F'(R)}{dt^2} + 9H\frac{dF'(R)}{dt} + \frac{1}{2}(-\rho + 3p)\,.\end{aligned} \tag{36}$$

Here C_ϕ is a constant. Then in the second line of Eq. (33), only (t,t) component does not vanish and behaves as a^{-3} and therefore the solution of Eq. (36) with $C_\phi \neq 0$ plays a role of the mimetic dark matter. On the other hand the (t,t) and (i,j)-components in (33) give the identical equation:

$$0 = \frac{d^2F'(R)}{dt^2} + 2H\frac{dF'(R)}{dt} - \left(\dot{H} + 3H^2\right)F'(R) + \frac{1}{2}F(R) + \frac{1}{2}p\,. \tag{37}$$

By combining (36) and (37), we obtain

$$0 = \frac{d^2F'(R)}{dt^2} - H\frac{dF'(R)}{dt} + 2\dot{H}F'(R) + \frac{1}{2}(p+\rho) + \frac{4C_\phi}{a^3}\,. \tag{38}$$

Mimetic F(R) gravity

When $C_\phi = 0$, the above equations reduce to those in the standard $F(R)$ gravity, or in other words, when $C_\phi \neq 0$, the equation and therefore the solutions are different from those in the standard $F(R)$ gravity. Lagrange multiplier constraint presentation:

Extended model. We may consider the following action of mimetic $F(R)$ gravity with scalar potential:

$$S = \int d^4x\sqrt{-g}\left(F\left(R\left(g_{\mu\nu}\right)\right) - V(\phi) + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1\right) + \mathcal{L}_{\mathrm{matter}}\right) . \qquad (39)$$

This action is of the sort of modified gravity with Lagrange multiplier constraint. Working with viable modified gravity one can reproduce the arbitrary evolution by changing scalar potential. This gives natural unification of inflation, dark matter and dark energy.

Singular evolution

The finite-time future singularities are classified as follows: Nojiri-Odintsov-Tsujikawa, PRD71,2005,063004.

- Type I ("Big Rip") : When $t \to t_s$, the scale factor diverges a, the effective energy density $\rho_{\rm eff}$, the effective pressure $p_{\rm eff}$ diverge, $a \to \infty$, $\rho_{\rm eff} \to \infty$, and $|p_{\rm eff}| \to \infty$. This type of singularity was presented in Caldwell-Kamionkowski-Weinberg,PRL91, 2003 where it was indicated that Rip occurs before entering singularity itself.
- Type II ("sudden") : When $t \to t_s$, the scale factor and the effective energy density is finite, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$ but the effective pressure diverges $|p_{\rm eff}| \to \infty$.
- Type III : When $t \to t_s$, the scale factor is finite, $a \to a_s$ but the effective energy density and the effective pressure diverge, $\rho_{\rm eff} \to \infty$, $|p_{\rm eff}| \to \infty$.
- Type IV : For $t \to t_s$, the scale factor, the effective energy density, and the effective pressure are finite, that is, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$, $|p_{\rm eff}| \to p_s$, but the higher derivatives of the Hubble rate $H \equiv \dot{a}/a$ diverge.

There is also possibility of change to decceleration in future, or approaching dS or infinite singularity (like Little Rip). It is interesting that future singularities may occur not only dark energy epoch but also at inflationary epoch: Barrow-Graham, PRD2015;Nojiri-Odintsov-Oikonomou,PRD91 (2015)084059.

Singular evolution

We consider the following action:

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R-\frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi-V(\phi)+L_{\mathrm{matter}}\right\}. \quad (40)$$

Choice of Hubble rate.In the case of the Type II and IV singularities, the Hubble rate $H(t)$ may be chosen in the following form:

$$H(t)=f_1(t)+f_2(t)\,(t_s-t)^\alpha\ . \quad (41)$$

Here $f_1(t)$ and $f_2(t)$ are smooth (differentiable) functions of t and α is a constant. If $0<\alpha<1$, there appears Type II singularity and if α is larger than 1 and not integer, there appears Type IV singularity. We first consider the simple case that $f_1(t)=0$ and $f_2(t)=f_0$ with a positive constant f_0. In the neighborhood of $t=t_s$, we find that,

$$\omega(\phi)=\frac{2\alpha f_0}{\kappa^2}\,(t_s-\phi)^{\alpha-1}\ ,\quad V(\phi)\sim-\frac{\alpha f_0}{\kappa^2}\,(t_s-\phi)^{\alpha-1}\ , \quad (42)$$

and we find

$$\varphi=-\frac{2\sqrt{2\alpha f_0}}{\kappa\,(\alpha+1)}\,(t_s-\phi)^{\frac{\alpha+1}{2}}\ , \quad (43)$$

Consequently, the scalar potential reads,

$$V(\varphi)\sim-\frac{\alpha f_0}{\kappa^2}\left\{-\frac{\kappa\,(\alpha+1)}{2\sqrt{2\alpha f_0}}\varphi\right\}^{\frac{2(\alpha-1)}{\alpha+1}}\ . \quad (44)$$

Singular evolution

Therefore, when the following condition holds true,

$$-2 < \frac{2(\alpha - 1)}{\alpha + 1} < 0 , \tag{45}$$

there occurs the Type II singularity. Accordingly, the Type IV singularity occurs when the following holds true,

$$0 < \frac{2(\alpha - 1)}{\alpha + 1} < 2 . \tag{46}$$

More examples maybe presented. Qualitatively: There could be three cases,

1. The Type IV singularity occurs during the inflationary era.
2. The inflationary era ends with the Type IV singularity.
3. The Type IV singularity occurs after the inflationary era.

Most realistically, we have second and third case, when we may get realistic inflation while universe survive transition over Type IV singularity. This scenario is also extended to F(R) gravity.Furthermore, one can get unification of singular inflation with dark energy via the same modified gravity. Singular inflation with exit thanks to singularity.

Unifying trace-anomaly driven inflation with cosmic acceleration in modified gravity. Bamba, Myrzakulov, Odintsov, Sebastiani, arXiv:1403.6649.

Trace anomaly reads (Duff 1994,Buchbinder-Odintsov-Shapiro 1992))

$$\langle T_{\mu}^{\mu} \rangle = \alpha \left(W + \frac{2}{3} \Box R \right) - \beta \mathcal{G} + \xi \Box R \,, \tag{47}$$

where $W = C^{\xi\sigma\mu\nu} C_{\xi\sigma\mu\nu}$ is the "square" of the Weyl tensor $C_{\xi\sigma\mu\nu}$ and $\mathcal{G}$ the Gauss-Bonnet topological invariant, given by

$$W = R^{\xi\sigma\mu\nu} R_{\xi\sigma\mu\nu} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \,, \quad \mathcal{G} = R^{\xi\sigma\mu\nu} R_{\xi\sigma\mu\nu} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \,, \tag{48}$$

The dimensionfull coefficients α, β, and ξ of the above expression are related to the number of conformal fields present in the theory. We introduce real scalar fields N_{S}, the Dirac (fermion) fields N_{F}, vector fields N_{V}, gravitons $N_2 (= 0\,,1)$, and higher-derivative conformal scalars N_{HD}. Then

$$\alpha = \frac{N_{\mathrm{S}} + 6N_{\mathrm{F}} + 12N_{\mathrm{V}} + 611 N_2 - 8N_{\mathrm{HD}}}{120(4\pi)^2} \,, \quad \beta = \frac{N_{\mathrm{S}} + 11N_{\mathrm{F}} + 62N_{\mathrm{V}} + 1411 N_2 - 28N_{\mathrm{HD}}}{360(4\pi)^2} \,, \tag{49}$$

If we exclude the contribution of gravitons and higher-derivative conformal scalars, we get

$$\alpha = \frac{1}{120(4\pi)^2} (N_{\mathrm{S}} + 6N_{\mathrm{F}} + 12N_{\mathrm{V}}) \,, \quad \beta = \frac{1}{360(4\pi)^2} (N_{\mathrm{S}} + 11N_{\mathrm{F}} + 62N_{\mathrm{V}}) \,, \quad \xi = -\frac{N_{\mathrm{V}}}{6(4\pi)^2} \,, \tag{50}$$

For $\mathcal{N}_{\mathrm{super}} = 4$ SU(N) super Yang-Mills (SYM) theory, we have $N_{\mathrm{S}} = 6N^2$, $N_{\mathrm{F}} = 2N^2$, and $N_{\mathrm{V}} = N^2$, where N is a very large number. Therefore, we obtain a relation among the numbers of scalars, spinors and vector fields.

$$\alpha = \beta = \frac{N^2}{64\pi^2} \,, \quad \xi = -\frac{N^2}{96\pi^2} \,. \tag{51}$$

Unifying trace-anomaly driven inflation with cosmic acceleration in modified gravity

Note that

$$\frac{2}{3}\alpha + \xi = 0\,, \tag{52}$$

and in principle the contribution of the $\Box R$ term to the conformal anomaly vanishes, but it could be reintroduced via a higher curvature term in the action (see below). Owing to the conformal anomaly, the classical Einstein equation is corrected as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = \kappa^2 \langle T_{\mu\nu} \rangle\,. \tag{53}$$

By taking the trace of the last equation (53), we derive

$$R = -\kappa^2 \langle T_\mu^\mu \rangle \equiv -\kappa^2 \left[\alpha \left(W + \frac{2}{3}\Box R \right) - \beta \mathcal{G} + \xi \Box R \right]\,. \tag{54}$$

Despite the fact that in Eq. (52), the coefficient of the $\Box R$ term is equal to zero, we can set it to any desired value by adding the finite R^2 counter term in the action. In the classical Einstein gravity, this additional term is necessary to exit from inflation (Starobinsky 1980). Concretely, by adding the following action

$$I = \frac{\gamma N^2}{192\pi^2} \int_{\mathcal{M}} d^4 x \sqrt{-g}\, R^2\,, \quad \gamma > 0\,, \tag{55}$$

Eq. (53) becomes (Dowker-Critchley 1976,Fishetti-Hartle-Hu 1979,Mamaev-Mostepanenko 1980, Starobinsky 1980)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = -\frac{\gamma N^2 \kappa^2}{48\pi^2} R R_{\mu\nu} + \frac{\gamma N^2 \kappa^2}{192\pi^2} R^2 g_{\mu\nu} + \frac{\gamma N^2 \kappa^2}{48\pi^2} \nabla_\mu \nabla_\nu R - \frac{\gamma N^2 \kappa^2}{48\pi^2} g_{\mu\nu} \Box R^2 + \kappa^2 \langle T_{\mu\nu} \rangle\,. \tag{56}$$

Account of F(R) gravity

The action is given by

$$I=\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R+2\kappa^2\tilde{\gamma}R^2+f(R)+2\kappa^2\mathcal{L}_{\mathrm{QC}}\right]\,,\quad \tilde{\gamma}\equiv\frac{\gamma N^2}{192\pi^2}\,, \tag{57}$$

where we have considered the R^2 term in the action with $\tilde{\gamma}$ as in (55) and we have added a correction given by a function $f(R)$ of the Ricci scalar. The field equations are

$$G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa^2\langle T_{\mu\nu}\rangle-4\tilde{\gamma}\kappa^2RR_{\mu\nu}+\tilde{\gamma}R^2\kappa^2g_{\mu\nu}+4\tilde{\gamma}\kappa^2\nabla_\mu\nabla_\nu R-4\tilde{\gamma}\kappa^2g_{\mu\nu}\Box R^2$$
$$-f_R(R)\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)+\frac{1}{2}g_{\mu\nu}[f(R)-Rf_R(R)]+(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box)f_R(R)\,, \tag{58}$$

The trace is described as

$$R=-\kappa^2\left(\alpha W-\beta G+\delta\Box R\right)-2f(R)+Rf_R(R)+3\Box f_R(R)\,, \tag{59}$$

where we have imposed the condition in Eq. (52) and introduced δ defined as

$$\delta\equiv-12\tilde{\gamma}=-\frac{\gamma N^2}{16\pi^2}\,,\quad \delta<0\,. \tag{60}$$

Here, $\gamma(>0)$ is a free parameter. The flat FLRW space-time

$$ds^2=-dt^2+a^2(t)\left(dx^2+dy^2+dz^2\right)\,, \tag{61}$$

The energy density ρ and pressure p of quantum corrections are represented as

$$\langle T_{00}\rangle=\rho\,,\quad \langle T_{ij}\rangle=p\,a(t)^2\delta_{ij}\,,\quad (i,j=1,2,3)\,. \tag{62}$$

Account of F(R) gravity

In the FLRW background, it follows from $(\mu,\nu)=(0,0)$ component and the trace part of $(\mu,\nu)=(i,j)$ of Eq. (58), we obtain the equations of motion

$$\frac{3}{\kappa^2}H^2 = \rho + \frac{1}{2\kappa^2}\left[Rf_R(R) - f(R) - 6H^2 f_R(R) - 6H\dot{f}_R(R)\right] \equiv \rho_{\rm eff}\,, \tag{63}$$

$$-\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right) = p + \frac{1}{2\kappa^2}\left[-Rf_R(R) + f(R) + (4\dot{H}+6H^2)f_R(R) + 4H\dot{f}_R(R) + 2\ddot{f}_R(R)\right] \equiv p_{\rm eff}\,. \tag{64}$$

In these equations, $\rho_{\rm eff}$ and $p_{\rm eff}$ are the effective energy density and pressure of the universe. The effective conservation law

$$\dot{\rho}_{\rm eff} + 3H\left(\rho_{\rm eff}+p_{\rm eff}\right) = 0\,. \tag{65}$$

The effective energy density is

$$\rho_{\rm eff} = \frac{\rho_0}{a^4} + 6\beta H^4 + \delta\left(18H^2\dot{H} + 6\ddot{H}H - 3\dot{H}^2\right) + \frac{1}{2\kappa^2}\left(Rf_R(R) - f(R) - 6H^2 f_R(R) - 6H\dot{f}_R(R)\right)\,, \tag{66}$$

where ρ_0 is the constant of integration. The effective pressure is

$$p_{\rm eff} = \frac{\rho_0}{3a^4} - \beta\left(6H^4 + 8H^2\dot{H}\right) - \delta\left(9\dot{H}^2 + 12H\ddot{H} + 2\dddot{H} + 18H^2\dot{H}\right) + \frac{1}{2\kappa^2}\left[-Rf_R(R) + f(R) + (4\dot{H}+6H^2)f_R(R) + 4H\dot{f}_R(R) + 2\ddot{f}_R(R)\right]\,. \tag{67}$$

In the expressions of $\rho_{\rm eff}$ in Eq. (66) and $p_{\rm eff}$ in Eq. (67), we can recognize the contributions from not only modified gravity but also quantum corrections. .

Trace-anomaly driven inflation in exponential gravity

Exponential $f(R)$ (Cognola-Elizalde-Nojiri-Odintsov-Zerbini 2007)

$$f(R) = -2\Lambda_{\rm eff}\left[1 - \exp\left(-\frac{R}{R_0}\right)\right] . \tag{68}$$

Indistinguishable from LCDM.
de Sitter solutions:

$$H_{\rm dS\pm}^2 = \frac{1}{4\beta\kappa^2}\left(1 \pm \sqrt{1 - \frac{8\zeta}{3}}\right) = \frac{2\pi M_{\rm Pl}^2}{N^2}\left(1 \pm \sqrt{1 - \frac{8\zeta}{3}}\right) ,$$
$$\Lambda_{\rm eff} = \frac{\zeta}{\beta\kappa^2} = \zeta\left[\frac{8\pi M_{\rm Pl}^2}{N^2}\right] , \quad 0 < \zeta < \frac{3}{8} . \tag{69}$$

There are two special solutions

$$H_{\rm dS}^2 = \frac{1}{2\beta\kappa^2} = \frac{4\pi M_{\rm Pl}^2}{N^2} , \quad \Lambda_{\rm eff} = 0 , \tag{70}$$

$$H_{\rm dS}^2 = \frac{1}{4\beta\kappa^2} = \frac{2\pi M_{\rm Pl}^2}{N^2} , \quad \Lambda_{\rm eff} = \frac{3}{8\beta\kappa^2} = \frac{3}{8}\left(\frac{8\pi M_{\rm Pl}^2}{N^2}\right) . \tag{71}$$

Stability of the de Sitter solutions We define the perturbations $\Delta H(t)$ as

$$H = H_{\rm dS\pm} + \Delta H(t) , \quad |\Delta H(t)| \ll 1 . \tag{72}$$

The solution is given by

$$\Delta H(t) = A_0 e^{\lambda_{1,2} t} , \quad \lambda_{1,2} = \frac{-3H_{\rm dS\pm} \pm \sqrt{9H_{\rm dS\pm}^2 + \frac{4}{\delta}\left(\frac{1}{\kappa^2} - 4H_{\rm dS\pm}^2\beta\right)}}{2} , \tag{73}$$

where A_0 is a constant.

Trace-anomaly driven inflation in exponential gravity

The de Sitter solutions of the model (68) are unstable (and adopted to describe the inflation) only if λ_1 (the eigenvalue with the positive sign in front of the square root) is real and positive, i.e.,

$$4\beta - \frac{1}{\kappa^2 H_{\mathrm{dS}\pm}^2} > 0\,, \quad 9H_{\mathrm{dS}\pm}^2 + \frac{4}{\delta}\left(\frac{1}{\kappa^2} - 4H_{\mathrm{dS}\pm}^2\beta\right) > 0\,. \tag{74}$$

Here, we have taken into account the fact that $\beta > 0$ and $\delta < 0$.

Dynamics of inflation

Given the unstable de Sitter solution $H_{\mathrm{dS}\pm}^2$ in , to analyze inflation occurring in the model in Eq. (68), we have to calculate the amplitude of the perturbations in Eq. (73).

At the time $t = 0$ when inflation starts, we have to set $\Delta H(t = 0) = 0$. The complete solution of this equation is given by the homogeneous part in Eq. (73) plus the contribute of modified gravity as follows

$$\Delta H(t) = A_0 \mathrm{e}^{\lambda_{1,2} t} - \frac{\mathrm{e}^{-R_{\mathrm{dS}}/R_0}\Lambda_{\mathrm{eff}}}{12 H_{\mathrm{dS}}\kappa^2}\left(\frac{R_{\mathrm{dS}}}{R_0} + 2\right)\left(\frac{1}{\kappa^2} - 4H_{\mathrm{dS}}^2\beta\right)^{-1}\,. \tag{75}$$

Thus, at $t = 0$, by putting $\Delta H(t = 0) = 0$, we can estimate the amplitude A_0 as

$$A_0 = -\frac{\mathrm{e}^{-R_{\mathrm{dS}}/R_0}\zeta}{12 H_{\mathrm{dS}}(\beta\kappa^2)}\left(\frac{R_{\mathrm{dS}}}{R_0} + 2\right)\left(1 - \frac{8}{3}\zeta\right)^{-1/2} < 0\,. \tag{76}$$

Here, we have considered only the unstable solution $H_{\mathrm{dS}} \equiv H_{\mathrm{dS}+}$ in Eq. (69).

The time at the end of inflation

$$t_{\mathrm{f}} \simeq \frac{R_{\mathrm{dS}}}{R_0\,\lambda_1}\,. \tag{77}$$

The number of e-folds $\mathcal{N}$ is

$$\mathcal{N} = \ln\left(\frac{a_{\mathrm{f}}}{a_{\mathrm{i}}}\right)\,, \tag{78}$$

and inflation is viable if $\mathcal{N} > 76$.

Trace-anomaly driven inflation in exponential gravity

For the model (68), by taking account of the fact that we have chosen $t_{\mathrm{i}} = 0$ and using Eq. (77), we acquire

$$\mathcal{N} \equiv H_{\mathrm{dS}} t_f = \frac{2R_{\mathrm{dS}}}{3R_0} \left[-1 + \sqrt{1 - \frac{16\beta}{9\delta} \left(\frac{\sqrt{1 - \frac{8}{3}\zeta}}{1 + \sqrt{1 - \frac{8}{3}\zeta}} \right)} \right]^{-1} . \tag{79}$$

By combining this relation, the expressions for β in Eq. (51) and δ in Eq. (60), and Eq. (79), we have

$$\mathcal{N} = \frac{2b}{3} \left[-1 + \sqrt{1 + \frac{4}{9\gamma} \left(\frac{\sqrt{1 - \frac{8}{3}\zeta}}{1 + \sqrt{1 - \frac{8}{3}\zeta}} \right)} \right]^{-1} . \tag{80}$$

Spectral index

The second time derivative of $a(t)$ is

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2 (1 - \epsilon) , \tag{81}$$

with the parameter ϵ. When the approximate de Sitter solution is realized, it has to be very small as

$$\epsilon = -\frac{\dot{H}}{H^2} \ll 1 . \tag{82}$$

Moreover, ϵ has to change very slowly. There is another parameter η, which has to also be very small as

$$|\eta| = \left| -\frac{\ddot{H}}{2H\dot{H}} \right| \equiv \left| \epsilon - \frac{1}{2\epsilon H}\dot{\epsilon} \right| \ll 1 . \tag{83}$$

These two parameters are the so-called slow-roll parameters.

Trace-anomaly driven inflation in exponential gravity

The amplitude of scalar-mode power spectrum of the primordial curvature perturbations at $k = 0.002\,\mathrm{Mpc}^{-1}$ is described as

$$\Delta_{\mathcal{R}}^2 = \frac{\kappa^2 H^2}{8\pi^2\epsilon}\,, \tag{84}$$

and the last cosmological data constrain the spectral index n_s and the tensor-to-scalar ratio r are given by (Mukhanov:1981),

$$n_\mathrm{s} = 1 - 6\epsilon + 2\eta\,, \quad r = 16\epsilon\,. \tag{85}$$

In the model (68), we find

$$\Delta_{\mathcal{R}}^2 = \frac{1}{32\pi^2\beta\epsilon}\left(1+\sqrt{1-\frac{8}{3}\zeta}\right) = \frac{2}{N^2\epsilon}\left(1+\sqrt{1-\frac{8}{3}\zeta}\right)\,, \tag{86}$$

The parameters ϵ and η read

$$\begin{aligned}\epsilon &\simeq -\frac{\Delta\dot{H}(t)}{H_{\mathrm{dS}}^2} = \frac{b^2}{\mathcal{N}^2}\left(-\frac{\delta}{4\beta}\right)\frac{\mathrm{e}^{(\lambda_1 t - b)}\zeta\,(b+2)}{\left(1-\frac{8}{3}\zeta\right)}\left(\frac{b}{3\mathcal{N}}+1\right) = \frac{b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{(\lambda_1 t - b)}\zeta\,(b+2)}{\left(1-\frac{8}{3}\zeta\right)}\left(\frac{b}{3\mathcal{N}}+1\right)\,,\\ \eta &= \epsilon - \frac{\dot{\epsilon}}{2\epsilon H_{\mathrm{dS}}} = \epsilon - \frac{\lambda_1}{2H_{\mathrm{dS}}} = \epsilon - \frac{b}{2\mathcal{N}}\,.\end{aligned} \tag{87}$$

During inflation, when $t \ll t_\mathrm{f}$, since $\mathcal{N} \gg 1$, we have

$$\epsilon \simeq \frac{b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1-\frac{8}{3}\zeta\right)} \ll 1\,, \quad |\eta| \simeq \left|-\frac{b}{2\mathcal{N}}\right| \ll 1\,. \tag{88}$$

Thus, the spectral index and the tensor-to-scalar ratio in Eq. (85) for the model (68) are derived as

$$n_\mathrm{s} = 1 - \frac{b}{\mathcal{N}} - \frac{6b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1-\frac{8}{3}\zeta\right)}\,, \quad r = \frac{16b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1-\frac{8}{3}\zeta\right)}\,. \tag{89}$$

Trace-anomaly driven inflation in exponential gravity

We mention the recent observations of the spectral index n_{s} as well as the tensor-to-scalar ratio r. The results observed by the Planck satellite are $n_{\mathrm{s}} = 0.9603 \pm 0.0073\,(68\%\,\mathrm{CL})$ and $r < 0.11\,(95\%\,\mathrm{CL})$. Since $b/\mathcal{N} \ll 1$ and $1 \ll b$, the constraints from the Planck satellite described above can be satisfied. For instance, for $b = 3$, $\zeta = 1/8$, and $\mathcal{N} = 76$, we have $n_{\mathrm{s}} \simeq 0.9601$ and $r = 1.20 \times 10^{-3}$.
On the other hand, the BICEP2 experiment has detected the B-mode polarization of the cosmic microwave background (CMB) radiation with the tensor to scalar ratio $r = 0.20^{+0.07}_{-0.05}\,(68\%\,\mathrm{CL})$, and also the case that r vanishes has been rejected at 7.0σ level.
For our model, even if the dependence of the tensor-to-scalar ratio on $\mathcal{N}^2$ makes it very small, we can play with a value of ζ close to $3/8$ in order to increase its value. For instance, with the choice $\zeta = 0.37125$, we can still describe the unstable de Sitter solution for $b > 1$, since $R_{\mathrm{dS}} \gg R_0$ and $f(R_{\mathrm{dS}}) \simeq -2\Lambda_{\mathrm{eff}}$. Thus, the number of e-folds $\mathcal{N}$ depends on γ only as in Eq. (80). Indeed, when we take the combination of the values of b and γ, e.g., $(b = 2, \gamma > 1.14)$, $(b = 3, \gamma > 0.76)$, and $(b = 4, \gamma > 0.57)$, and so on, we obtain $\mathcal{N} > 76$.
For example, if $\mathcal{N} = 76$, for $b = 2$, 3 and 4, we acquire $r = 0.22$, 0.23, and 0.18, respectively.
Thus, unification of realistic inflation with viable dark energy era occurs in exponential F(R) gravity with account of quantum effects (trace anomaly). This is in full accord with first discovery of such unification proposed in Nojiri-Odintsov2003.

Anti-evaporation of SdS BHs in F(R) theory

L.Sebastiani, D. Momeni, R.Myrzakulov, S.D.Odintsov, arXiv:1305.4231

Nariai metric in the cosmological patch with $R_0 = 4\Lambda$ and cosmological time t given by $\tau = \arccos\,[\cosh t]^{-1}$ reads

$$ds^2 = -\frac{1}{\Lambda \cos^2 \tau}\left(-d\tau^2 + dx^2\right) + \frac{1}{\Lambda} d\Omega^2 \,, \tag{90}$$

$-\pi/2 < \tau < \pi/2$. $F(R)$-gravity admits such a metric as the limiting case of the Schwarzshild-de Sitter solution under the condition

$$2F(R_0) = R_0 F_R(R_0)\,. \tag{91}$$

Perturbations around the Nariai space-time are described by

$$ds^2 = \mathrm{e}^{2\rho(x,\tau)}\left(-d\tau^2 + dx^2\right) + \mathrm{e}^{-2\varphi(x,\tau)} d\Omega^2 \,, \quad \rho = -\ln\left[\sqrt{\Lambda}\cos\tau\right] + \delta\rho \,, \quad \varphi = \ln\sqrt{\Lambda} + \delta\varphi \,. \tag{92}$$

From the field equations of $F(R)$-gravity one finds

$$\frac{1}{\alpha\cos^2\tau}\left[2(2\alpha-1)\delta\varphi\right] - 3\delta\ddot{\varphi} + 3\delta\varphi'' = 0\,, \quad \alpha = \frac{2\Lambda F_{RR}(R_0)}{F'(R_0)}\,, \tag{93}$$

and

$$\delta R \equiv 4\Lambda\left(-\delta\rho + \delta\varphi\right) + \Lambda\cos^2\tau\left(2\delta\ddot{\rho} - 2\delta\rho'' - 4\delta\ddot{\varphi} + 4\delta\varphi''\right) = 2\frac{F_R(R_0)}{F_{RR}(R_0)}\delta\varphi\,. \tag{94}$$

Equation (93) can be used to study the evolution of $\varphi(\tau, x)$. In principle, one may insert the result in (94) in order to obtain $\rho(\tau, x)$. However, the radius of the Nariai black hole depends on $\varphi(\tau, x)$ only, so that we will limit our analysis to Eq. (93).

Anti-evaporation of SdS BHs in F(R) theory

Horizon perturbations.
The position of the horizon moves on the one-sphere S_1 and it is located in the correspondence of $\nabla\delta\varphi \cdot \nabla\delta\varphi = 0$. For a black hole located at $x = x_0$, the horizon is defined as

$$r_0(\tau)^{-2} = \mathrm{e}^{2\varphi(\tau, x_0)} = \frac{1 + \delta\varphi(x_0, \tau)}{\Lambda} \,. \tag{95}$$

Therefore, evaporation/anti-evaporation correspond to increasing/decreasing values of $\delta\varphi(\tau)$ on the horizon.
Following [J. C. Niemeyer and R. Bousso, Phys. Rev. D **62** (2000) 023503 [gr-qc/0004004]] we can decompose the two-sphere radius of Nariai solution into Fourier modes on the S_1 sphere, namely

$$\delta\varphi(x, t) = \epsilon \sum_{n=1}^{+\infty} \left(A_n(\tau)\cos[nx] + B_n(\tau)\sin[nx]\right) \,, \quad 1 \gg \epsilon > 0 \,. \tag{96}$$

Here, ϵ is assumed to be positive and small. From Eq. (93) we get

$$\delta\varphi(x, t) = \epsilon \sum_{n=1}^{\infty} P_\nu^\mu(\xi) \left[a_n \cos(nx) + b_n \sin(nx)\right] \,, \quad \xi = \sin\tau \,, \tag{97}$$

with

$$\mu = \sqrt{\frac{2(2\alpha - 1)}{3\alpha}} \,, \quad \nu = -\frac{1}{2} \pm \sqrt{n^2 + \frac{1}{4}} \,, \quad \alpha = \frac{2\Lambda F_{RR}(R_0)}{F'(R_0)} \,. \tag{98}$$

Above, $P_\nu^\mu(\xi)$ are the Legendre polynomials regular on the boundary $\xi = 0$ (i.e. $t = 0$) and the unknown coefficients $\{a_n, b_n\}$ can in principle be obtained by using the initial boundary conditions at $\xi = 0$.

Anti-evaporation of SdS BHs in F(R) theory

By using this formalism, we can study the stability/unstability of Nariai solutions in $F(R)$-gravity for different modes of $\delta\varphi(x,t)$. For $n = 1$ one has near to $\xi = 1$ (i.e. $t \to +\infty$):

- When μ is real

$$P_\nu^\mu(\xi) \simeq (1-\xi)^{-\frac{\mu}{2}} \left[\frac{2^{\mu/2}}{\Gamma(1-\mu)} - \frac{2^{\mu/2}(\mu-\mu^2+2\nu(1+\nu))}{4\Gamma(2-\mu)}(1-\xi) + \mathcal{O}(1-\xi)^2 \right] . \tag{99}$$

  This is the case of α real and $1/2 < \alpha$ or $\alpha < 0$, for example models like $F(R) = R + \gamma R^m$. The Legendre polynomial and therefore the Nariai horizon diverge. We have anti-evaporation (or evaporation if $\epsilon < 0$ from the beginning).

- When μ is complex number

$$P_\nu^{i|\mu|}(\xi) \simeq (1-\xi)^{-\frac{i|\mu|}{2}} \left[\frac{2^{\frac{i|\mu|}{2}}}{\Gamma(1-i|\mu|)} - \frac{2^{\frac{i|\mu|}{2}}(1-\xi)}{4\Gamma(2-i|\mu|)}(|\mu|(i+|\mu|)+2\nu(\nu+1)) + \mathcal{O}(1-\xi)^2) \right] . \tag{100}$$

  This is the case of $0 < \alpha < 1/2$, for example models like $F(R) = R - 2\Lambda(1 - e^{R/R^*})$. The Legendre polynomial and therefore the Nariai horizon do not diverge. Solution is stable, we can have only transient evaporation/antievaporation.

Stable neutron stars from $f(R)$ gravity

A.Astashenok,S. Capozziello and S.D. Odintsov,arXiv:1309.1978

It is convenient to write function $f(R)$ as

$$f(R) = R + \alpha h(R), \tag{101}$$

The field equations are

$$(1 + \alpha h_R)G_{\mu\nu} - \frac{1}{2}\alpha(h - h_R R)g_{\mu\nu} - \alpha(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)h_R = \frac{8\pi G}{c^4}T_{\mu\nu}\,. \tag{102}$$

Spherically symmetric metric with two independent functions of radial coordinate:

$$ds^2 = -e^{2\phi}c^2dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{103}$$

The energy–momentum tensor $T_{\mu\nu} = \mathrm{diag}(e^{2\phi}\rho c^2, e^{2\lambda}P, r^2P, r^2\sin^2\theta P)$, where ρ is the matter density and P is the pressure. The components of the field equations are

$$\begin{aligned} \frac{-8\pi G}{c^2}\rho &= -r^{-2} + e^{-2\lambda}(1 - 2r\lambda')r^{-2} + \alpha h_R(-r^{-2} + e^{-2\lambda}(1 - 2r\lambda')r^{-2}) \\ &\quad -\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha[h'_R r^{-1}(2 - r\lambda') + h''_R], \end{aligned} \tag{104}$$

$$\begin{aligned} \frac{8\pi G}{c^4}P &= -r^{-2} + e^{-2\lambda}(1 + 2r\phi')r^{-2} + \alpha h_R(-r^{-2} + e^{-2\lambda}(1 + 2r\phi')r^{-2}) \\ &\quad -\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha h'_R r^{-1}(2 + r\phi'), \end{aligned} \tag{105}$$

where prime denotes derivative with respect to radial distance, r.

Stable neutron stars from $f(R)$ gravity

For the exterior solution, we assume a Schwarzschild solution. For this reason, it is convenient to define the change of variable

$$e^{-2\lambda} = 1 - \frac{2GM}{c^2 r} \,. \tag{106}$$

The value of parameter M on the surface of a neutron star can be considered as a gravitational star mass. Useful relation

$$\frac{GdM}{c^2 dr} = \frac{1}{2}\left[1 - e^{-2\lambda}(1 - 2r\lambda')\right) \,, \tag{107}$$

.

The hydrostatic condition of equilibrium can be obtained from the Bianchi identities

$$\frac{dP}{dr} = -(\rho + P/c^2)\frac{d\phi}{dr} \,, . \tag{108}$$

The second TOV equation can be obtained by substitution of the derivative $d\phi/dr$ from (108) in Eq.(105). The dimensionless variables

$$M = mM_\odot \,, \quad r \to r_g r, \quad \rho \to \rho M_\odot / r_g^3, \quad P \to pM_\odot c^2/r_g^3, \quad R \to R/r_g^2.$$

Here $M_\odot$ is the Sun mass and $r_g = GM_\odot/c^2 = 1.47473$ km. Eqs. (104), (105) can be rewritten as

$$\left(1 + \alpha r_g^2 h_R + \frac{1}{2}\alpha r_g^2 h_R' r\right)\frac{dm}{dr} = 4\pi\rho r^2 - \frac{1}{4}\alpha r^2 r_g^2\left(h - h_R R - 2\left(1 - \frac{2m}{r}\right)\left(\frac{2h_R'}{r} + h_R''\right)\right) \,, \tag{109}$$

$$8\pi p = -2\left(1 + \alpha r_g^2 h_R\right)\frac{m}{r^3} - \left(1 - \frac{2m}{r}\right)\left(\frac{2}{r}(1 + \alpha r_g^2 h_R) + \alpha r_g^2 h_R'\right)(\rho + p)^{-1}\frac{dp}{dr} - \tag{110}$$

$$-\frac{1}{2}\alpha r_g^2\left(h - h_R R - 4\left(1 - \frac{2m}{r}\right)\frac{h_R'}{r}\right) \,,$$

where $' = d/dr$.

Stable neutron stars from $f(R)$ gravity

For $\alpha = 0$, Eqs. (109), (110) reduce to

$$\frac{dm}{dr} = 4\pi\tilde{\rho}r^2 \tag{111}$$

$$\frac{dp}{dr} = -\frac{4\pi pr^3 + m}{r(r-2m)}\left(\tilde{\rho} + p\right), \tag{112}$$

i.e. to ordinary dimensionless TOV equations. These equations can be solved numerically for a given EoS $p = f(\rho)$ and initial conditions $m(0) = 0$ and $\rho(0) = \rho_c$.
For non-zero α, one needs the third equation for the Ricci curvature scalar. The trace of field Eqs. (102) gives the relation

$$3\alpha\Box h_R + \alpha h_R R - 2\alpha h - R = -\frac{8\pi G}{c^4}(-3P + \rho c^2). \tag{113}$$

In dimensionless variables, we have

$$3\alpha r_g^2 \left(\left(\frac{2}{r} - \frac{3m}{r^2} - \frac{dm}{rdr} - \left(1 - \frac{2m}{r}\right)\frac{dp}{(\rho+p)dr}\right)\frac{d}{dr} + \left(1 - \frac{2m}{r}\right)\frac{d^2}{dr^2}\right) h_R$$
$$+ \alpha r_g^2 h_R R - 2\alpha r_g^2 h - R = -8\pi(\rho - 3p). \tag{114}$$

We need to add the EoS for matter inside star to the Eqs. (109), (110), (114). Standard polytropic EoS $p \sim \rho^\gamma$ works, although a more realistic EoS has to take into account different physical states for different regions of the star and it is more complicated.
Perturbative solution. For a perturbative solution the density, pressure, mass and curvature can be expanded as

$$p = p^{(0)} + \alpha p^{(1)} + \ldots, \quad \rho = \rho^{(0)} + \alpha\rho^{(1)} + \ldots, \tag{115}$$
$$m = m^{(0)} + \alpha m^{(1)} + \ldots, \quad R = R^{(0)} + \alpha R^{(1)} + \ldots,$$

where functions $\rho^{(0)}$, $p^{(0)}$, $m^{(0)}$ and $R^{(0)}$ satisfy to standard TOV equations assumed at zeroth order. Terms containing h_R are assumed to be of first order in the small parameter α, so all such terms should be evaluated at $\mathcal{O}(\alpha)$ order.

Stable neutron stars from $f(R)$ gravity

For $m = m^{(0)} + \alpha m^{(1)}$, the following equation

$$\frac{dm}{dr} = 4\pi\rho r^2 - \alpha r^2\left(4\pi\rho^{(0)}h_R + \frac{1}{4}\left(h - h_R R\right)\right) + \frac{1}{2}\alpha\left(\left(2r - 3m^{(0)} - 4\pi\rho^{(0)}r^3\right)\frac{d}{dr} + r(r - 2m^{(0)})\frac{d^2}{dr^2}\right) \tag{116}$$

for pressure $p = p^{(0)} + \alpha p^{(1)}$

$$\frac{r-2m}{\rho+p}\frac{dp}{dr} = 4\pi r^2 p + \frac{m}{r} - \alpha r^2\left(4\pi p^{(0)}h_R + \frac{1}{4}\left(h - h_R R\right)\right) - \alpha\left(r - 3m^{(0)} + 2\pi p^{(0)}r^3\right)\frac{dh_R}{dr}. \tag{117}$$

The Ricci curvature scalar, in terms containing h_R and h, has to be evaluated at $\mathcal{O}(1)$ order, i.e.

$$R \approx R^{(0)} = 8\pi(\rho^{(0)} - 3p^{(0)})\,. \tag{118}$$

We can consider various EoS for the description of the behavior of nuclear matter at high densities. For example the SLy and FPS equation have the same analytical representation:

$$\zeta = \frac{a_1 + a_2\xi + a_3\xi^3}{1 + a_4\xi}f(a_5(\xi - a_6)) + (a_7 + a_8\xi)f(a_9(a_{10} - \xi)) + \tag{119}$$
$$+(a_{11} + a_{12}\xi)f(a_{13}(a_{14} - \xi)) + (a_{15} + a_{16}\xi)f(a_{17}(a_{18} - \xi)),$$

where

$$\zeta = \log(P/\text{dyncm}^{-2})\,, \qquad \xi = \log(\rho/\text{gcm}^{-3})\,, \qquad f(x) = \frac{1}{\exp(x)+1}\,.$$

The coefficients a_i for SLy and FPS EoS are different.
Neutron star with a quark core. The quark matter can be described by the very simple EoS:

$$p_Q = a(\rho - 4B), \tag{120}$$

where a is a constant and the parameter B can vary from $\sim$ 60 to 90 Mev/fm^3.

Stable neutron stars from $f(R)$ gravity

For quark matter with massless strange quark, it is $a = 1/3$. We consider $a = 0.28$ corresponding to $m_s = 250$ Mev. For numerical calculations, Eq. (120) is used for $\rho \geq \rho_{tr}$, where ρ_{tr} is the transition density for which the pressure of quark matter coincides with the pressure of ordinary dense matter. For example for FPS equation, the transition density is $\rho_{tr} = 1.069 \times 10^{15}$ g/cm^3 ($B = 80$ Mev/fm^3), for SLy equation $\rho_{tr} = 1.029 \times 10^{15}$ g/cm^3 ($B = 60$ Mev/fm^3).

Model 1.

$$f(R) = R + \beta R(\exp(-R/R_0) - 1), \tag{121}$$

We can assume, for example, $R = 0.5 r_g^{-2}$. For $R << R_0$ this model coincides with quadratic model of $f(R)$ gravity.
For neutron stars models with quark core, there is no significant differences with respect to General Relativity. For a given central density, the star mass grows with α. The dependence is close to linear for $\rho \sim 10^{15}$g/cm^3. For the piecewise equation of state (FPS case for $\rho < \rho_{tr}$) the maximal mass grows with increasing α. For $\beta = -0.25$, the maximal mass is $1.53 M_\odot$, for $\beta = 0.25$, $M_{max} = 1.59 M_\odot$ (in General Relativity, it is $M_{max} = 1.55 M_\odot$). With an increasing β, the maximal mass is reached at lower central densities. Furthermore, for $dM/d\rho_c < 0$, there are no stable star configurations. A similar situation is observed in the SLy case but mass grows with β more slowly.
For the simplified EoS (119), other interesting effects can occur. For $\beta \sim -0.15$ at high central densities ($\rho_c \sim 3.0 - 3.5 \times 10^{15}$g/cm^3), we have the dependence of the neutron star mass from radius and from central density. For $\beta < 0$ for high central densities we have the stable star configurations ($dM/d\rho_c > 0$).

Stable neutron stars from $f(R)$ gravity

For example the measurement of mass of the neutron star PSR J1614-2230 with $1.97 \pm 0.04\ M_\odot$ provides a stringent constraint on any $M - R$ relation. The model with SLy equation is more interesting: in the context of model (121), the upper limit of neutron star mass is around $2M_\odot$ and there is second branch of stability star configurations at high central densities. This branch describes observational data better than the model with SLy EoS in GR.
Possibility of a stabilization mechanism in $f(R)$ gravity which leads to the existence of stable neutron stars which are more compact objects than in General Relativity. Cubic model.

$$f(R) = R + \alpha R^2(1 + \gamma R)\,. \qquad (122)$$

Let $|\gamma R| \sim \mathcal{O}(1)$ for large R and $\alpha R^2(1 + \gamma R) << R$. For small masses, the results coincide with R^2 model. For $\gamma = -10$ (in units r_g^2) the maximal mass of neutron star at high densities $\rho > 3.7 \times 10^{15}$ g/cm^3 is nearly $1.88M_\odot$ and radius is about ~ 9 km (SLy equation). For $\gamma = -20$ the maximal mass is $1.94M_\odot$ and radius is about ~ 9.2 km . In the GR, for SLy equation, the minimal radius of neutron stars is nearly 10 km. Therefore such a model of $f(R)$ gravity can give rise to neutron stars with smaller radii than in GR. Therefore such theory can describe (assuming only the SLy equation), the existence of peculiar neutron stars with mass $\sim 2M_\odot$ (the measured mass of PSR J1614-2230) and compact stars ($R \sim 9$ km) with masses $M \sim 1.6 - 1.7M_\odot$.
For smaller values of γ the minimal neutron star mass (and minimal central density at which stable stars exist) on second branch of stability decreases.
It is interesting to note that for negative and sufficiently large values of ϵ, the maximal limit of neutron star mass can exceed the limit in General Relativity for given EoS (the stable stars exist for higher central densities). Therefore some EoS which ruled out by observational constraints in GR can describe real star configurations in frames of such model of gravity. One has to note that the upper limit in this model of gravity is achieved for smaller radii than in GR for acceptable EoS.

$f(\mathcal{G})$ gravity: General properties

Topological Gauss-Bonnet invariant:

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} , \tag{123}$$

is added to the action of the Einstein gravity. One starts with the following action:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(\mathcal{G}) + \mathcal{L}_{\rm matter}\right) . \tag{124}$$

Here, $\mathcal{L}_{\rm matter}$ is the Lagrangian density of matter. The variation of the metric $g_{\mu\nu}$:

$$\begin{aligned} 0 =& \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu}_{\rm matter} + \frac{1}{2}g^{\mu\nu}f(\mathcal{G}) - 2f'(\mathcal{G})RR^{\mu\nu} \\ & + 4f'(\mathcal{G})R^{\mu}{}_{\rho}R^{\nu\rho} - 2f'(\mathcal{G})R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4f'(\mathcal{G})R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^\mu\nabla^\nu f'(\mathcal{G})\right)R \\ & - 2g^{\mu\nu}\left(\nabla^2 f'(\mathcal{G})\right)R - 4\left(\nabla_\rho\nabla^\mu f'(\mathcal{G})\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f'(\mathcal{G})\right)R^{\mu\rho} \\ & + 4\left(\nabla^2 f'(\mathcal{G})\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\mu\rho\nu\sigma} . \end{aligned} \tag{125}$$

The first FRW equation:

$$0 = -\frac{3}{\kappa^2}H^2 - f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 + \rho_{\rm matter} . \tag{126}$$

Here $\mathcal{G}$ has the following form:

$$\mathcal{G} = 24\left(H^2\dot{H} + H^4\right) . \tag{127}$$

the FRW-like equations (fluid description):

$$\rho^{\mathcal{G}}_{\rm eff} = \frac{3}{\kappa^2}H^2 , \quad p^{\mathcal{G}}_{\rm eff} = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right) . \tag{128}$$

$f(\mathcal{G})$ gravity: General properties

Here,

$$\rho_{\mathrm{eff}}^{\mathcal{G}} \equiv -f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 + \rho_{\mathrm{matter}} ,$$
$$p_{\mathrm{eff}}^{\mathcal{G}} \equiv f(\mathcal{G}) - \mathcal{G}f'(\mathcal{G}) + \frac{2\mathcal{G}\dot{\mathcal{G}}}{3H} f''(\mathcal{G}) + 8H^2\ddot{\mathcal{G}}f''(\mathcal{G}) + 8H^2\dot{\mathcal{G}}^2 f'''(\mathcal{G}) + p_{\mathrm{matter}} . \tag{129}$$

When $\rho_{\mathrm{matter}} = 0$, Eq. (126) has a de Sitter universe solution where H, and therefore $\mathcal{G}$, are constant. For $H = H_0$, with a constant H_0, Eq. (126) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24H_0^4 f'\left(24H_0^4\right) - f\left(24H_0^4\right) . \tag{130}$$

As an example, we consider the model

$$f(\mathcal{G}) = f_0 \left|\mathcal{G}\right|^{\beta} , \tag{131}$$

with constants f_0 and β. Then, the solution of Eq. (130) is given by

$$H_0^4 = \frac{1}{24\left(8\left(n-1\right)\kappa^2 f_0\right)^{\frac{1}{\beta-1}}} . \tag{132}$$

No matter and GR. Eq. (126) reduces to

$$0 = \mathcal{G}f'(\mathcal{G}) - f(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 . \tag{133}$$

If $f(\mathcal{G})$ behaves as (131), assuming

$$a = \begin{cases} a_0 t^{h_0} & \text{when } h_0 > 0 \text{ (quintessence)} \\ a_0 \left(t_s - t\right)^{h_0} & \text{when } h_0 < 0 \text{ (phantom)} \end{cases} , \tag{134}$$

one obtains

$$0 = (\beta - 1)\, h_0^6 \left(h_0 - 1\right)\left(h_0 - 1 + 4\beta\right) . \tag{135}$$

$f(\mathcal{G})$ gravity: General properties

As $h_0 = 1$ implies $\mathcal{G} = 0$, one may choose

$$h_0 = 1 - 4\beta \,, \tag{136}$$

and Eq. (3) gives

$$w_{\rm eff} = -1 + \frac{2}{3(1-4\beta)} \,. \tag{137}$$

Therefore, if $\beta > 0$, the universe is accelerating ($w_{\rm eff} < -1/3$), and if $\beta > 1/4$, the universe is in a phantom phase ($w_{\rm eff} < -1$). Thus, we are led to consider the following model:

$$f(\mathcal{G}) = f_i \left|\mathcal{G}\right|^{\beta_i} + f_l \left|\mathcal{G}\right|^{\beta_l} \,, \tag{138}$$

where it is assumed that

$$\beta_i > \frac{1}{2} \,, \quad \frac{1}{2} > \beta_l > \frac{1}{4} \,. \tag{139}$$

Then, when the curvature is large, as in the primordial universe, the first term dominates, compared with the second term and the Einstein term, and it gives

$$-1 > w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_i)} > -\frac{5}{3} \,. \tag{140}$$

On the other hand, when the curvature is small, as is the case in the present universe, the second term in (138) dominates compared with the first term and the Einstein term and yields

$$w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_l)} < -\frac{5}{3} \,. \tag{141}$$

Therefore, theory (138) can produce a model that is able to describe inflation and the late-time acceleration of the universe in a unified manner.

$f(\mathcal{G})$ gravity: General properties

The action (124) can be rewritten by introducing the auxiliary scalar field ϕ as,

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - V(\phi) - \xi(\phi)\mathcal{G}\right] . \tag{142}$$

By variation over ϕ, one obtains

$$0 = V'(\phi) + \xi'(\phi)\mathcal{G} , \tag{143}$$

which could be solved with respect to ϕ as

$$\phi = \phi(\mathcal{G}) . \tag{144}$$

By substituting the expression (144) into the action (142), we obtain the action of $f(\mathcal{G})$ gravity, with

$$f(\mathcal{G}) = -V\left(\phi(\mathcal{G})\right) + \xi\left(\phi(\mathcal{G})\right)\mathcal{G} . \tag{145}$$

Assuming a spatially-flat FRW universe and the scalar field ϕ to depend only on t, we obtain the field equations:

$$0 = -\frac{3}{\kappa^2}H^2 + V(\phi) + 24H^3\frac{d\xi(\phi(t))}{dt} , \tag{146}$$

$$\begin{aligned} 0 =& \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) - V(\phi) - 8H^2\frac{d^2\xi(\phi(t))}{dt^2} \\ & - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} - 16H^3\frac{d\xi(\phi(t))}{dt} . \end{aligned} \tag{147}$$

Combining the above equations, we obtain

$$\begin{aligned} 0 =& \frac{2}{\kappa^2}\dot{H} - 8H^2\frac{d^2\xi(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} + 8H^3\frac{d\xi(\phi(t))}{dt} \\ =& \frac{2}{\kappa^2}\dot{H} - 8a\frac{d}{dt}\left(\frac{H^2}{a}\frac{d\xi(\phi(t))}{dt}\right) , \end{aligned} \tag{148}$$

$f(\mathcal{G})$ gravity: General properties

which can be solved with respect to $\xi(\phi(t))$ as

$$\xi(\phi(t)) = \frac{1}{8}\int^t dt_1 \frac{a(t_1)}{H(t_1)^2} W(t_1)\,, \quad W(t) \equiv \frac{2}{\kappa^2}\int^t \frac{dt_1}{a(t_1)} \dot{H}(t_1)\,. \tag{149}$$

Combining (146) and (149), the expression for $V(\phi(t))$ follows:

$$V(\phi(t)) = \frac{3}{\kappa^2}H(t)^2 - 3a(t)H(t)W(t)\,. \tag{150}$$

As there is a freedom of redefinition of the scalar field ϕ, we may identify t with ϕ. Hence, we consider the model where $V(\phi)$ and $\xi(\phi)$ can be expressed in terms of a single function g as

$$\begin{aligned} V(\phi) =& \frac{3}{\kappa^2} g'(\phi)^2 - 3g'(\phi)\,\mathrm{e}^{g(\phi)}U(\phi)\,, \\ \xi(\phi) =& \frac{1}{8}\int^\phi d\phi_1 \frac{\mathrm{e}^{g(\phi_1)}}{g'(\phi_1)^2}U(\phi_1)\,, \\ U(\phi) \equiv& \frac{2}{\kappa^2}\int^\phi d\phi_1 \mathrm{e}^{-g(\phi_1)} g''(\phi_1)\,. \end{aligned} \tag{151}$$

By choosing $V(\phi)$ and $\xi(\phi)$ as (151), one can easily find the following solution for Eqs.(146) and (147):

$$a = a_0 \mathrm{e}^{g(t)} \quad (H = g'(t))\,. \tag{152}$$

Therefore one can reconstruct $F(G)$ gravity to generate arbitrary expansion history of the universe. Thus, we reviewed the modified Gauss-Bonnet gravity and demonstrated that it may naturally lead to the unified cosmic history, including the inflation and dark energy era.

String-inspired model and scalar-Einstein-Gauss-Bonnet gravity

Stringy gravity:

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{2} + \mathcal{L}_\phi + \mathcal{L}_c + \dots\right] , \tag{153}$$

where ϕ is the dilaton, $\mathcal{L}_\phi$ is the Lagrangian of ϕ, and $\mathcal{L}_c$ expresses the string curvature correction terms,

$$\mathcal{L}_\phi = -\partial_\mu\phi\partial^\mu\phi - V(\phi) , \quad \mathcal{L}_c = c_1\alpha' \mathrm{e}^{2\frac{\phi}{\phi_0}} \mathcal{L}_c^{(1)} + c_2\alpha'^2 \mathrm{e}^{4\frac{\phi}{\phi_0}} \mathcal{L}_c^{(2)} + c_3\alpha'^3 \mathrm{e}^{6\frac{\phi}{\phi_0}} \mathcal{L}_c^{(3)} , \tag{154}$$

where $1/\alpha'$ is the string tension, $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$, and $\mathcal{L}_c^{(3)}$ express the leading-order (Gauss-Bonnet term $\mathcal{G}$ in (123)), the second-order, and the third-order curvature corrections, respectively:

$$\mathcal{L}_c^{(1)} = \Omega_2 , \quad \mathcal{L}_c^{(2)} = 2\Omega_3 + R^{\mu\nu}_{\alpha\beta} R^{\alpha\beta}_{\lambda\rho} R^{\lambda\rho}_{\mu\nu} , \quad \mathcal{L}_c^{(3)} = \mathcal{L}_{31} - \delta_H \mathcal{L}_{32} - \frac{\delta_B}{2}\mathcal{L}_{33} . \tag{155}$$

Here, δ_B and δ_H take the value of 0 or 1 and

$$\begin{aligned}
\Omega_2 &= \mathcal{G} , \\
\Omega_3 &\propto \epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'} R_{\mu\nu}{}^{\mu'\nu'} R_{\rho\sigma}{}^{\rho'\sigma'} R_{\tau\eta}{}^{\tau'\eta'} , \\
\mathcal{L}_{31} &= \zeta(3) R_{\mu\nu\rho\sigma} R^{\alpha\nu\rho\beta}\left(R^{\mu\gamma}{}_{\delta\beta} R_{\alpha\gamma}{}^{\delta\sigma} - 2R^{\mu\gamma}{}_{\delta\alpha} R_{\beta\gamma}{}^{\delta\sigma}\right) , \\
\mathcal{L}_{32} &= \frac{1}{8}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 + \frac{1}{4} R_{\mu\nu}{}^{\gamma\delta} R_{\gamma\delta}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} - \frac{1}{2} R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma} R^{\mu}{}_{\sigma\gamma\delta} R_{\rho}{}^{\nu\gamma\delta} - R_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta}{}^{\rho\nu} R_{\rho\sigma}{}^{\gamma\delta} R_{\gamma} \\
\mathcal{L}_{33} &= \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 - 10 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\sigma} R_{\sigma\gamma\delta\rho} R^{\beta\gamma\delta\rho} - R_{\mu\nu\alpha\beta} R^{\mu\nu\rho}{}_{\sigma} R^{\beta\sigma\gamma\delta} R_{\delta\gamma\rho}{}^{\alpha}
\end{aligned} \tag{156}$$

The correction terms are different depending on the type of string theory; the dependence is encoded in the curvature invariants and in the coefficients (c_1, c_2, c_3) and δ_H, δ_B, as follows,

- For the Type II superstring theory: $(c_1, c_2, c_3) = (0, 0, 1/8)$ and $\delta_H = \delta_B = 0$.
- For the heterotic superstring theory: $(c_1, c_2, c_3) = (1/8, 0, 1/8)$ and $\delta_H = 1, \delta_B = 0$.
- For the bosonic superstring theory: $(c_1, c_2, c_3) = (1/4, 1/48, 1/8)$ and $\delta_H = 0, \delta_B = 1$.

String-inspired model and scalar-Einstein-Gauss-Bonnet gravity

The starting action is:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \xi(\phi)\mathcal{G}\right] . \tag{157}$$

Field equations:

$$\begin{aligned} 0 = & \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}g^{\mu\nu}\left(-V(\phi) + \xi(\phi)\mathcal{G}\right) \\ & - 2\xi(\phi)RR^{\mu\nu} - 4\xi(\phi)R^\mu{}_\rho R^{\nu\rho} - 2\xi(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4\xi(\phi)R^{\mu\rho\nu\sigma}R_{\rho\sigma} \\ & + 2\left(\nabla^\mu\nabla^\nu\xi(\phi)\right)R - 2g^{\mu\nu}\left(\nabla^2\xi(\phi)\right)R - 4\left(\nabla_\rho\nabla^\mu\xi(\phi)\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu\xi(\phi)\right)R^{\mu\rho} \\ & + 4\left(\nabla^2\xi(\phi)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\rho\sigma} + 4\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\mu\rho\nu\sigma} . \end{aligned} \tag{158}$$

FRW eq.:

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{1}{2}\dot{\phi}^2 + V(\phi) + 24H^3\frac{d\xi(\phi(t))}{dt} , \tag{159}$$

$$\begin{aligned} 0 = & \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) + \frac{1}{2}\dot{\phi}^2 - V(\phi) - 8H^2\frac{d^2\xi(\phi(t))}{dt^2} \\ & - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} - 16H^3\frac{d\xi(\phi(t))}{dt} . \end{aligned} \tag{160}$$

Scalar equation

$$0 = \ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \xi'(\phi)\mathcal{G} . \tag{161}$$

String-inspired model and scalar-Einstein-Gauss-Bonnet gravity

In particular when we consider the following string-inspired model,

$$V = V_0 \mathrm{e}^{-\frac{2\phi}{\phi_0}} , \quad \xi(\phi) = \xi_0 \mathrm{e}^{\frac{2\phi}{\phi_0}} , \tag{162}$$

the de Sitter space solution follows:

$$H^2 = H_0^2 \equiv - \frac{\mathrm{e}^{-\frac{2\varphi_0}{\phi_0}}}{8\xi_0\kappa^2} , \quad \phi = \varphi_0 . \tag{163}$$

Here, φ_0 is an arbitrary constant. If φ_0 is chosen to be larger, the Hubble rate $H = H_0$ becomes smaller. Then, if $\xi_0 \sim \mathcal{O}(1)$, by choosing $\varphi_0/\phi_0 \sim 140$, the value of the Hubble rate $H = H_0$ is consistent with the observations. The model (162) also has another solution:

$$\begin{aligned} H &= \frac{h_0}{t} , \quad \phi = \phi_0 \ln \frac{t}{t_1} \qquad \text{when } h_0 > 0 , \\ H &= -\frac{h_0}{t_s - t} , \quad \phi = \phi_0 \ln \frac{t_s - t}{t_1} \qquad \text{when } h_0 < 0 . \end{aligned} \tag{164}$$

Here, h_0 is obtained by solving the following algebraic equations:

$$0 = -\frac{3h_0^2}{\kappa^2} + \frac{\phi_0^2}{2} + V_0 t_1^2 - \frac{48\xi_0 h_0^3}{t_1^2} , \quad 0 = (1 - 3h_0)\,\phi_0^2 + 2V_0 t_1^2 + \frac{48\xi_0 h_0^3}{t_1^2} (h_0 - 1) . \tag{165}$$

Eqs. (165) can be rewritten as

$$V_0 t_1^2 = -\frac{1}{\kappa^2 (1 + h_0)} \left\{ 3h_0^2 (1 - h_0) + \frac{\phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right\} , \tag{166}$$

$$\frac{48\xi_0 h_0^2}{t_1^2} = -\frac{6}{\kappa^2 (1 + h_0)} \left(h_0 - \frac{\phi_0^2 \kappa^2}{2} \right) . \tag{167}$$

The arbitrary value of h_0 can be realized by properly choosing V_0 and ξ_0. With the appropriate choice of V_0 and ξ_0, we can obtain a negative h_0 and, therefore, the effective EoS parameter (3) is less than -1, $w_{\mathrm{eff}} < -1$, which corresponds to the effective phantom.

$F(R)$ bigravity

Non-linear massive gravity (with non-dynamical background metric) was extended to the ghost-free construction with the dynamical metric (Hassan et al).
The convenient description of the theory gives bigravity or bimetric gravity which contains two metrics (symmetric tensor fields). One of two metrics is called physical metric while second metric is called reference metric.
Next is $F(R)$ bigravity which is also ghost-free theory. We introduce four kinds of metrics, $g_{\mu\nu}$, $g^{\mathrm{J}}_{\mu\nu}$, $f_{\mu\nu}$, and $f^{\mathrm{J}}_{\mu\nu}$. The physical observable metric $g^{\mathrm{J}}_{\mu\nu}$ is the metric in the Jordan frame. The metric $g_{\mu\nu}$ corresponds to the metric in the Einstein frame in the standard $F(R)$ gravity and therefore the metric $g_{\mu\nu}$ is not physical metric. In the bigravity theories, we have to introduce another reference metrics or symmetric tensor $f_{\mu\nu}$ and $f^{\mathrm{J}}_{\mu\nu}$. The metric $f_{\mu\nu}$ is the metric corresponding to the Einstein frame with respect to the curvature given by the metric $f_{\mu\nu}$. On the other hand, the metric $f^{\mathrm{J}}_{\mu\nu}$ is the metric corresponding to the Jordan frame.
The starting action is given by

$$\begin{aligned} S_{\mathrm{bi}} =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\ &+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1}f}\right) . \end{aligned} \tag{169}$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ and $R^{(f)}$ is the scalar curvature for $f_{\mu\nu}$. M_{eff} is defined by

$$\frac{1}{M_{\mathrm{eff}}^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2} . \tag{170}$$

Furthermore, tensor $\sqrt{g^{-1}f}$ is defined by the square root of $g^{\mu\rho} f_{\rho\nu}$, that is, $\left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \rho} \left(\sqrt{g^{-1}f}\right)^{\rho}_{\ \nu} = g^{\mu\rho} f_{\rho\nu}$.

$F(R)$ bigravity

For general tensor $X^\mu{}_\nu$, $e_n(X)$'s are defined by

$$\begin{aligned}
& e_0(X) = 1\,, \quad e_1(X) = [X]\,, \quad e_2(X) = \tfrac{1}{2}([X]^2 - [X^2])\,, \\
& e_3(X) = \tfrac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3])\,, \\
& e_4(X) = \tfrac{1}{24}([X]^4 - 6[X]^2[X^2] + 3[X^2]^2 + 8[X][X^3] - 6[X^4])\,, \\
& e_k(X) = 0 \quad \text{for } k > 4\,.
\end{aligned} \tag{171}$$

Here $[X]$ expresses the trace of arbitrary tensor $X^\mu{}_\nu$: $[X] = X^\mu{}_\mu$. In order to construct the consistent $F(R)$ bigravity, we add the following terms to the action (169):

$$S_\varphi = -M_g^2 \int d^4x \sqrt{-\det g}\left\{\frac{3}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right\} + \int d^4x \mathcal{L}_{\rm matter}\left(e^\varphi g_{\mu\nu}, \Phi_i\right)\,, \tag{172}$$

$$S_\xi = -M_f^2 \int d^4x \sqrt{-\det f}\left\{\frac{3}{2} f^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + U(\xi)\right\}\,. \tag{173}$$

By the conformal transformations $g_{\mu\nu} \to e^{-\varphi} g^{\rm J}_{\mu\nu}$ and $f_{\mu\nu} \to e^{-\xi} f^{\rm J}_{\mu\nu}$, the total action $S_F = S_{\rm bi} + S_\varphi + S_\xi$ is transformed as

$$\begin{aligned}
S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\rm J}}\left\{e^{-\xi} R^{{\rm J}(f)} - e^{-2\xi} U(\xi)\right\} \\
& + 2m^2 M_{\rm eff}^2 \int d^4x \sqrt{-\det g^{\rm J}} \sum_{n=0}^{4} \beta_n e^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{{\rm J}\,-1} f^{\rm J}}\right) \\
& + M_g^2 \int d^4x \sqrt{-\det g^{\rm J}}\left\{e^{-\varphi} R^{{\rm J}(g)} - e^{-2\varphi} V(\varphi)\right\} \\
& + \int d^4x \mathcal{L}_{\rm matter}\left(g^{\rm J}_{\mu\nu}, \Phi_i\right)\,.
\end{aligned} \tag{174}$$

$F(R)$ bigravity

The kinetic terms for φ and ξ vanish. By the variations with respect to φ and ξ as in the case of convenient $F(R)$ gravity, we obtain

$$\begin{aligned} 0 =& 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2 \right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} \right. \\ & \left. + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} , \qquad (175) \end{aligned}$$

$$0 = -\, 2m^2 M_{\mathrm{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} . \qquad (176)$$

The Eqs. (175) and (176) can be solved algebraically with respect to φ and ξ as

$$\varphi = \varphi \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \right)$$

and

$$\xi = \xi \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \right)$$

. Substituting above φ and ξ into (174), one gets $F(R)$ bigravity:

$$\begin{aligned} S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} F^{(f)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \right) \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \\ & + M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} F^{\mathrm{J}(g)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \right) + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g_{\mu\nu}^{\mathrm{J}}, \Phi_i \right) , \qquad (177) \end{aligned}$$

$F(R)$ bigravity

$$F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(g)} \right.$$
$$\left. - \mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} V\left(\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)\right)\right\}, \tag{178}$$

$$F^{(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(f)} \right.$$
$$\left. - \mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} U\left(\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)\right)\right\}. \tag{179}$$

Note that it is difficult to solve Eqs. (175) and (176) with respect to φ and ξ explicitly. Therefore, it might be easier to define the model in terms of the auxiliary scalars φ and ξ as in (174).

$F(R)$ bigravity: Cosmological Reconstruction and Cosmic Acceleration

Let us consider the cosmological reconstruction program. For simplicity, we start from the minimal case

$$
\begin{aligned}
S_{\mathrm{bi}} =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f}\right) . \qquad (180)
\end{aligned}
$$

In order to evaluate $\delta\sqrt{g^{-1}f}$, two matrices M and N, which satisfy the relation $M^2 = N$ are taken. Since $\delta M M + M\delta M = \delta N$, one finds

$$
\mathrm{tr}\,\delta M = \frac{1}{2}\mathrm{tr}\,\left(M^{-1}\delta N\right) . \qquad (181)
$$

For a while, we consider the Einstein frame action (180) with (172) and (173) but matter contribution is neglected. Then by the variation over $g_{\mu\nu}$, we obtain

$$
\begin{aligned}
0 =& M_g^2 \left(\frac{1}{2}g_{\mu\nu}R^{(g)} - R^{(g)}_{\mu\nu}\right) + m^2 M_{\mathrm{eff}}^2 \left\{ g_{\mu\nu}\left(3 - \mathrm{tr}\,\sqrt{g^{-1}f}\right) \right. \\
&\left. + \frac{1}{2} f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\nu} + \frac{1}{2} f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\mu} \right\} \\
&+ M_g^2 \left[\frac{1}{2}\left(\frac{3}{2}g^{\rho\sigma}\partial_\rho\varphi\partial_\sigma\varphi + V(\varphi)\right) g_{\mu\nu} - \frac{3}{2}\partial_\mu\varphi\partial_\nu\varphi\right] . \qquad (182)
\end{aligned}
$$

On the other hand, by the variation over $f_{\mu\nu}$, we get

$$
\begin{aligned}
0 =& M_f^2 \left(\frac{1}{2}f_{\mu\nu}R^{(f)} - R^{(f)}_{\mu\nu}\right) + m^2 M_{\mathrm{eff}}^2 \sqrt{\det\left(f^{-1}g\right)} \left\{ -\frac{1}{2} f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu} \right. \\
&\left. - \frac{1}{2} f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\mu} + \det\left(\sqrt{g^{-1}f}\right) f_{\mu\nu} \right\} + M_f^2 \left[\frac{1}{2}\left(\frac{3}{2}f^{\rho\sigma}\partial_\rho\xi\partial_\sigma\xi + U(\xi)\right) f_{\mu\nu} - \frac{3}{2}\partial_\mu\xi\partial_\nu\xi\right] . \\
&\qquad (183)
\end{aligned}
$$

$F(R)$ bigravity: Cosmological Reconstruction and Cosmic Acceleration

We should note that $\det\sqrt{g}\det\sqrt{g^{-1}f} \neq \sqrt{\det f}$ in general. The variations of the scalar fields φ and ξ are given by

$$0 = -3\Box_g\varphi + V'(\varphi)\,, \quad 0 = -3\Box_f\xi + U'(\xi)\,. \tag{184}$$

Here $\Box_g$ ($\Box_f$) is the d'Alembertian with respect to the metric g (f). By multiplying the covariant derivative ∇_g^μ with respect to the metric g with Eq. (182) and using the Bianchi identity $0 = \nabla_g^\mu\left(\frac{1}{2}g_{\mu\nu}R^{(g)} - R^{(g)}_{\mu\nu}\right)$ and Eq. (184), we obtain

$$0 = -\, g_{\mu\nu}\nabla_g^\mu\left(\mathrm{tr}\sqrt{g^{-1}f}\right) + \frac{1}{2}\nabla_g^\mu\left\{f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \nu} + f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}_{\quad\ \mu}\right\}\,. \tag{185}$$

Similarly by using the covariant derivative ∇_f^μ with respect to the metric f, from (183), we obtain

$$0 = \nabla_f^\mu\left[\sqrt{\det\left(f^{-1}g\right)}\left\{-\frac{1}{2}\left(\sqrt{g^{-1}f}\right)^{-1\nu}_{\quad\ \sigma}g^{\sigma\mu} - \frac{1}{2}\left(\sqrt{g^{-1}f}\right)^{-1\mu}_{\quad\ \sigma}g^{\sigma\nu} + \det\left(\sqrt{g^{-1}f}\right)f^{\mu\nu}\right\}\right]\,. \tag{186}$$

In case of the Einstein gravity, the conservation law of the energy-momentum tensor depends from the Einstein equation. It can be derived from the Bianchi identity. In case of bigravity, however, the conservation laws of the energy-momentum tensor of the scalar fields are derived from the scalar field equations. These conservation laws are independent of the Einstein equation. The Bianchi identities give equations (185) and (186) independent of the Einstein equation.
We now assume the FRW universes for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ and use the conformal time t for the universe with metric $g_{\mu\nu}$:

$$\begin{aligned} ds_g^2 &= \sum_{\mu,\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu = a(t)^2\left(-dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2\right)\,, \\ ds_f^2 &= \sum_{\mu,\nu=0}^{3} f_{\mu\nu}dx^\mu dx^\nu = -c(t)^2dt^2 + b(t)^2\sum_{i=1}^{3}\left(dx^i\right)^2\,. \end{aligned} \tag{187}$$

$F(R)$ bigravity: Cosmological Reconstruction and Cosmic Acceleration

Then (t, t) component of (182) gives

$$0 = -3M_g^2 H^2 - 3m^2 M_{\text{eff}}^2 \left(a^2 - ab\right) + \left(\frac{3}{4}\dot{\varphi}^2 + \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 , \tag{188}$$

and (i, j) components give

$$\begin{aligned} 0 =& M_g^2 \left(2\dot{H} + H^2\right) + m^2 M_{\text{eff}}^2 \left(3a^2 - 2ab - ac\right) \\ &+ \left(\frac{3}{4}\dot{\varphi}^2 - \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 . \end{aligned} \tag{189}$$

Here $H = \dot{a}/a$. On the other hand, (t, t) component of (183) gives

$$0 = -3M_f^2 K^2 + m^2 M_{\text{eff}}^2 c^2 \left(1 - \frac{a^3}{b^3}\right) + \left(\frac{3}{4}\dot{\xi}^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 , \tag{190}$$

and (i, j) components give

$$\begin{aligned} 0 =& M_f^2 \left(2\dot{K} + 3K^2 - 2LK\right) + m^2 M_{\text{eff}}^2 \left(\frac{a^3 c}{b^2} - c^2\right) \\ &+ \left(\frac{3}{4}\dot{\xi}^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 . \end{aligned} \tag{191}$$

Here $K = \dot{b}/b$ and $L = \dot{c}/c$. Both of Eq. (185) and Eq. (186) give the identical equation:

$$cH = bK \text{ or } \frac{c\dot{a}}{a} = \dot{b} . \tag{192}$$

If $\dot{a} \neq 0$, we obtain $c = a\dot{b}/\dot{a}$. On the other hand, if $\dot{a} = 0$, we find $\dot{b} = 0$, that is, a and b are constant and c can be arbitrary.

$F(R)$ bigravity: Cosmological Reconstruction and Cosmic Acceleration

We now redefine scalars as $\varphi = \varphi(\eta)$ and $\xi = \xi(\zeta)$ and identify η and ζ with the conformal time t, $\eta = \zeta = t$. Hence, one gets

$$\omega(t) M_g^2 = -4M_g^2\left(\dot{H} - H^2\right) - 2m^2 M_{\mathrm{eff}}^2 (ab - ac)\,, \tag{193}$$

$$\tilde{V}(t) a(t)^2 M_g^2 = M_g^2\left(2\dot{H} + 4H^2\right) + m^2 M_{\mathrm{eff}}^2 (6a^2 - 5ab - ac)\,, \tag{194}$$

$$\sigma(t) M_f^2 = -4M_f^2\left(\dot{K} - LK\right) - 2m^2 M_{\mathrm{eff}}^2\left(-\frac{c}{b} + 1\right)\frac{a^3 c}{b^2}\,, \tag{195}$$

$$\tilde{U}(t) c(t)^2 M_f^2 = M_f^2\left(2\dot{K} + 6K^2 - 2LK\right) + m^2 M_{\mathrm{eff}}^2\left(\frac{a^3 c}{b^2} - 2c^2 + \frac{a^3 c^2}{b^3}\right)\,. \tag{196}$$

Here

$$\omega(\eta) = 3\varphi'(\eta)^2\,, \quad \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right)\,, \quad \sigma(\zeta) = 3\xi'(\zeta)^2\,, \quad \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right)\,. \tag{197}$$

Therefore for arbitrary $a(t)$, $b(t)$, and $c(t)$ if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy Eqs. (193-196), the cosmological model with given $a(t)$, $b(t)$ and $c(t)$ evolution can be reconstructed. Following this technique we presented number of inflationary and/or dark energy models as well as unified inflation-dark energy cosmologies. The method is general and maybe applied to more exotic and more complicated cosmological solutions.

What is the next?

What is the next? So far F(R) gravity which also admits extensions as HL or massive gravity is considered to be the best: simplest formulation, ghost-free, easy emergence of unified description for the universe evolution, friendly passing of cosmological bounds and local tests, absence of singularities in some versions(Bamba-Nojiri-Odintsov 2007), possibility of easy further modifications. More deep cosmological tests are necessary to understand if this is final phenomenological theory of universe and how it is related with yet to be constructed QG!